Alasmarya Islamic University

Faculty of Science

Department of Mathematics

Separation Axioms In Fibrewise Ideal Topological Spaces

A thesis submitted in partial fulfillment of the requirements for the degree of master in mathematics

By

Najia Mohammed Ebrahem Elsherany

Supervised by

PROF. Mabruk Ali Younes Sola

Zliten

2020-2022

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ

وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَدْخِلْنِي بِرَحْمَتِكَ فِي عِبَادِكَ الصَّالِحِينَ ﴾

صَدَقَ اللَّهُ العَظِيمُ

سورة النمل الآية (19)

ACKNOWLEDCEMENTS

First, all gratitude and thanks is due to Allah almighty, who inspired me to bring – forh to light the material covered in this thesis.

I would, like to express my deep thanks and gratitude to prof. Dr. Mabruk Ali Younes Sola, professor of Mathematics, Faculty of Science, Tripoli university, for suggesting the problem, tremendous efforts, valuable help, unlimited patience and continuous supervision through preparing this thesis.

I wish to express my sincere thanks and gratitude to prof. Dr. Rmadan jehaima, professor of Mathematics, Faculty of Science, Mesrata university, for his nice guidance and encouragement.

I wish to express my sincere thanks and gratitude to Dr. Mohamed hamoda, assist prof of Mathematics, Faculty of Science ALasmarya Islamic university, for his nice guidance and encouragement.

I would like to thank all my friends who helped me dring the writing of this thesis, especially, Dr. Najah Salem.

Finally, I'm appreciative to my mother, my brothers, and my sisters for their support, patience and continuous encouragement.

DEDICATION

TO THE MEMORY

OF

MY FATHER

AND

MY SISTER

Abstract

If (X, τ) is a topological space, I is an ideal on X and $\tau^*(I)$ the topology finer than τ induced by the ideal I, then for any topological space B with a continuous map $p:X\rightarrow B$ we call $(X, \tau^*(I))$ the fibrewise ideal topological space over B with fiber subspaces $\{ X_b : b \in B\}$; where $X_b = P^{-1}(b)$ for all $b \in B$.

The aim of this thesis is to define separation axioms in fibrewise ideal topological spaces and to study some of their basic properties. Also we discuss the main concepts, the important results in the topic including the relationship between these axioms and with the known separation axioms.

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Introduction

Fibrewise topological spaces theory, presented in the recent 20 years, as a new branch of mathematics developed on the basis of General Topolog, Algebraic Topology. It is associated with differential geometry, lie groups and dynamical systems theory. From the perspective of category theory, it is in the higher category of general topological spaces, so the discussion of new properties and characteristics of the variety of fibre topological spaces has more important significance [8].

Fibrewise topology can be thought of as the topology of continuous families of spaces or maps.

A continuous map P: $E \rightarrow B$, now is called fibrewise topological space over B, and $E(b) = P^{-1}(b)$ can be thought of as a continuous family of spaces,b $\in$ B. The parameter space B is called the base space, $E(b) = P^{-1}(b)$ is the fibre over b [13].

Ideals in topological spaces have been considered since 1930. In 1990 once again Jankovie and Hamlett, initiated the application of topological ideals in the generalization of most fundamental properties in general topology, they studied separation axioms using the concept of ideals in topological spaces [17].

Introduction

In 2018, fibrewise ideals was defined on fibrewise topology, and studied some of its properties, and considered fibrewise local function for a fibrewise space X over B using fibrewise ideal on X.

The aim of this thesis is to define the separation axioms in fibrewise ideal topological spaces, and discuss some of their properties.

This thesis consists of four chapters:

Chapter one is an introductory considered as a background for the material included in this thesis. It contains basic concepts, definitions, properties and some theorems of topological spaces.

Chapter two consists of two sections. Section one introduces the concepts of fibrewise topology. Also, it defines the fibrewise direct product of fibrewise topology. Section two studies separation axioms in fibrewise topological spaces, some examples are given, and interseted properties.

Chapter three consists of three sections. Section one studies the definition of fibrewise ideal and its properties. Section two defines the fibrewise local function for a fibrewise space X over B. Section three restricts the definition of a local function on each fibre X_b over $b \in B$ using a fibrewise ideal, and studies its properties.

Chapter four defines separation axioms in fibrewise ideal topological spaces, introduces some examples and studies the properties of fibrewise ideal spaces and the relationships between them using fibrewise maps. In addition, it proves several new results concering it.

Chapter one

Preliminary Concepts

This chapter is designed to give the preliminary concepts needed to the other chapters, and it is containing five sections.

1-1 Topological spaces:-

Definition: 1-1-1 [20]

Let $X \neq \emptyset$ and $\tau \subseteq P(X)$

(the collection of all subsets of X) such that the following three axioms hold:

1- $\emptyset \in \tau$ and $X \in \tau$.

2- If $G_\alpha \in \tau$, for $\alpha \in \Lambda$, then $\cup \{G_\alpha : \alpha \in \Lambda\} \in \tau$.

3- If $G_i \in \tau$, (i = 1,2,3,…...n), then $\bigcap_{i=1}^{n} G_i \in \tau$

then τ is a topology on X, the elements of τ are called open sets, and the pair (X, τ) is called a topological space.

Examples: 1-1-2

1- Let X be any non-empty set,

$\tau = \{X, \emptyset\}$, then (X,τ) is a topological space, τ called the trivial topology on X, and (X,τ) is called a trivial space.

2- Let X be any non-empty set, τ = Ƥ(X) (the power set of X), then τ a topology on X called The discrete topology on X, and (X,τ), is called a discrete space.

3- Let $X = \mathbb{R}, \tau = \{U \subseteq \mathbb{R}$: if $x \in U$, then there is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset U\}$. Then τ is a topology on X, called the usual topology and $(\mathbb{R}, \tau)$ is called the usual space.

4- Let $X = \mathbb{R}^n$, $\tau = \{U \subseteq \mathbb{R}^n$: if $x \in U$ then there is $\varepsilon > 0$ with $B_\varepsilon(x) \subseteq U\}$. Where $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\}$ then τ is a topology on $\mathbb{R}^n$ called the usual topology.

Definition: 1- 1- 3 [20]

Let (X, τ) be a topological space, then F is said to be closed set in X iff it's complement is an open set.

Example: 1-1-4

Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, then the closed sets in X are $X, \emptyset, \{a, c, d\}$, $\{b, c, d\}, \{c, d\}$

Theorem: 1-1-5 [20]

Let(X,τ) be a topological space. Then:

i. $\emptyset$ and X are closed sets

ii. If $A_\alpha \subset X$ is closed, for $\alpha \in \Lambda$, then $\bigcap A_\alpha \{A_\alpha : \alpha \in \Lambda\}$ is closed.

iii. If $A_i \subset X$ is closed $(i = 1,2,3 \dots n)$, then $\bigcup_{i=1}^{n} A_i$ is closed.

Proof:

i. $\emptyset$ and X are closed, since their respective complements X and $\emptyset$ are open.

ii. Let $A_\alpha \subset X$ be closed, for $\alpha \in \Lambda$. This implies that $(X - A_\alpha) \in \tau$ for $\alpha \in \Lambda$. Also $X - \bigcap\{A_\alpha : \alpha \in \Lambda\} = \bigcup\{X - A_\alpha : \alpha \in \Lambda\}$. Since $X - A_\alpha \in \tau$ for $\alpha \in \Lambda$, $\bigcup\{X - A_\alpha : \alpha \in \Lambda\} \in \tau$. Thus $\{A_\alpha , \alpha \in \Lambda\}$ is closed

iv. Let $A_i \subset X$ be closed ($i = 1,2,3\ldots n$). This implies that $X - A_i \in \tau$ ($i = 1,2,3\ldots n$). Also , $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(x - A_i)$ Since $X - A_i \in \tau$ ($i = 1,2,3\ldots n$) , $\bigcap_{i=1}^{n}(x - A_i) \in \tau$ .Thus $\bigcup_{i=1}^{n} A_i$ is closed.■

Definition: 1-1-6 [20]

Let (X, τ) be a topological space and $\emptyset \neq A \subset X$. The subspace (relative) topology on A is $\tau_A = \{A \cap G : G \in \tau\}$. (A, τ_A) is called a subspace of (X, τ)

Example: 1-1-7

1- Let (X,τ) be a topological space and $\tau = P(X)$. Let $\emptyset \neq Y \subseteq X$, then $\tau_Y = P(Y)$ is the discrete topology on Y.

2- If X is infinite set, and τ the finite complement space (the cofinite topology) i-e, $\tau = \{A: X \setminus A \text{ is finite set}\} \cup \emptyset$. Let Y be a finite subset of X, then τ_Y is the discrete topology on Y.

Theorem: 1- 1-8

Let (Y, τ_Y) be a subspace of a topological space (Y, τ_Y), then E is a closed set of Y iff there exists a closed set F of X, such that E = F$\cap$ Y.

proof:

Let E be a closed subset of Y, then Y\E is an open subset of Y. So there exists an open set A of X such that Y\ E = A $\cap$ Y .Thus E =$A^c \cap$ Y , that is there exists a closed set F = A^c, with E = F $\cap$ Y. Conversely, let F be a closed subset of X, such that $Y \cap F = E$, then $Y \cap F^c =$ Y\E . Therefore Y \ E is an open subset of Y. So E is a closed set in Y.

1-2 Elementry Concepts:

Definition: 1-2-1[20],[16]

Let (X, τ) be a topological space and A $\subset$ X. The closure of A is the set, $\overline{A} = \bigcap\{F \subset X : A \subseteq F \text{ and } F \text{ is closed}\}$.

Theorem: 1-2-2 [20]

Let (X, τ) be a topological space and A$\subset$ X. Then x $\in \overline{A}$ iff x $\in$ G $\in \tau$ implies $G \cap A \neq \emptyset$

Proof:

Let x $\in \overline{A}$ and x $\in$ G $\in \tau$. Assume that $G \cap A = \emptyset$. This implies that A $\subset$ X\G and X\G is closed. Hence x $\in$ X\G, and X\G is a closed set containing A, a contradiction. Conversely, suppose that $x \in G \in \tau$ implies $G \cap A \neq \emptyset$. Assume that x $\notin \overline{A}$ .Then there is a closed subset F of X such that $A \subset F$ and $x \notin F$. Hence $x \in X \backslash F \in \tau$ and $(X \setminus F) \cap A = \emptyset$.

Contradiction. ■

Theorem: 1-2-3 [20]

Let (X,τ)be a topological space and A, B are subsets of X. Then the following statements are true:

i. $\overline{\emptyset} = \emptyset$

ii. $A \subset \overline{A}$

iii. $\overline{(\overline{A})} = \overline{A}$

iv. $\overline{AUB} = \overline{A} \cup \overline{B}$

Proof:

i. $\emptyset = \bigcap\{F \subset X : \emptyset \subseteq F \text{ and F is closed}\} = \emptyset$, since $\emptyset$ is a closed set.

ii. Since $A \subset \bigcap\{F \subset X : A \subseteq F \text{ and is closed}\}$, so $A \subseteq \overline{A}$.

iii. $\overline{A} = \{F \subset X : \overline{A} \subseteq F \text{ and F is closed}\}$ is closed, and $\overline{A} \subset \overline{\overline{A}}$, Hence

$\overline{(\overline{A})} \cap \{F \subset X : \overline{A} \subset F \text{ and F is a closed}\} = \overline{A}$

iv. Since $A \subset \overline{A}$ and $B \subset \overline{B}$ by (ii), we have $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$, is a closed set, we have $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Also $\overline{A \cup B}$ is a closed superset of A and B. Hence $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, which imply that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ■

Theorem: 1-2-4[7]

Let X be a non-empty set and,

$cl^*: P(X) \to P(X)$ a map with the properties

i) $A \subset cl^*(A)$ for any $A \subset X$.

ii) $cl^*(cl^*(A)) = cl^*(A)$ for any $A \subset X$.

iii) $cl^*(A \cup B) = cl^*(A) \cup cl^*(B)$ for any two subsets A , B of X .

iv) $cl^*(\emptyset) = \emptyset$

Then cl^* is called a kuratowski closure operator and there is a topology on X such that A is a closed set in X iff $cl^*(A) = A$. The family $\tau^* = \{ [cl^*(A)]^c : A \in P(X)\}$ is a topology on X, for which $cl^*(A) = \overline{A}$ for any $A \subseteq X$.

Proof:

To show that τ^* is a topology on X

a) $\emptyset \in \tau^*$ because by **i**) we have $X \subseteq cl^*(X)$ and so $cl^*(X) = X$.

Hence $[cl^*(X)]^c = (X)^c = \emptyset \in \tau^*$.

Also, $X \in \tau^*$ since $cl^*(\emptyset) = \emptyset$ and so $[cl^*(\emptyset)]^c = \emptyset^c = X \in \tau^*$.

b) Let A, B$\in \tau^*$, then since

$[cl^*(A)]^c \cap [cl^*(B)]^c = [cl^*(A) \cup cl^*(B)]^c = [cl^*(A \cup B)]^c \in \tau^*$.

c) Let $A_\alpha \in \tau^*$ for all $\alpha \in \Lambda$.

Let $S = \cup [cl^*(A_\alpha)]^c$ we are going to show that

$S = [cl^*(K)]^c$ for some $K \subset X$. Because

$S = [\cap \, cl^*(A_\alpha)]^c$, we find $S^c = \underset{\alpha \in \Lambda}{\cap} cl^*(A_\alpha) \subset$

$cl^*(A_\alpha)$ for each α. So by (**iv**)and(**iii**)$cl^*(S^c) \subset$

$cl^*(A_\alpha)$ for all α and therefor $cl^*(S^c) \subset \underset{\alpha \in \Lambda}{\cap} cl^*(A_\alpha) =$

S^c. Then by (**ii**) $cl^*(S^c) = S^c$ and so $S = [\, cl^*(S^c)]^c \in$

τ^*. Therefore τ^* is a topology on X. To show $cl^*(A) =$

$\overline{A}$. First, $\overline{A} \subset cl^*(A)$, since $cl^*(A)$is closed in τ^* and $A \subset$

$cl^*(A)$, so we have, $\overline{A} \subset \overline{cl^*(A)} =$

$cl^*(A)$. Also since $(\overline{A})^c$ is open in τ^*, we have $\overline{A} =$

$cl^*(B)$for some $B \subset X$, using (**iv**)and

$A \subset \overline{A}$ we find that $cl^*(A) \subset cl^*(\overline{A}) = cl^* \, cl^*(B) = cl^*(B) = \overline{A}$. ■

Definition: 1-2-5 [16] , [7]

Let A⊂X. The interior (Int (A)) of A is the largest open set contained in A, that is Int (A) = ∪ {U: U open and U ⊂ A}

Examples: 1-2-6

1) In $\mathbb{R}$ with the usual topology if A= {0,1}, then Int (A) = ∅

2) In $\mathbb{R}^2$ with the usual topology , $A = \left\{(x, y): y = \sin(1/x), 0 < x \leq 1\right\}$

Then Int (A) = ∅

Definition: 1-2-7 [7]

Let A ⊂ X. The boundry d(A) of A is $\overline{A} \cap \overline{A^c}$

Examples: 1-2-8

Let A= [0,1] ⊂ ℝ with the usual topology, then $d(A) = \{0,1\}$

Definition: 1-2-9 [18]

A subset A of a space (X , τ) is said to be compact if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of A, there exist a finite subcover $\{U_{\alpha i}\}_{i=1}^{n}$ of A, such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha i}$

1-3 Open, Closed Maps And Continuous Maps:

Definition: 1-3-1 [7]

A map *f*: X → Y is called open (resp.closed) if the image of each open (resp . closed) set in X is open (resp.closed) set in Y

Examples: 1-3-2

1) Let *f:* $\mathbb{R}^2 \to \mathbb{R}$ be the projection mapping defined as $f(x_1, x_2) = x_1$, then *f* is an open map. But not closed.

2) If Y is a discrete space (all subsets are open) then every function *f :* X→ Y is both open and closed .

Theorem: 1–3-3 [7]

$f : X \to Y$ is a closed map iff $\overline{f(A)} \subseteq f(\overline{A})$ for each set A ⊂ X .

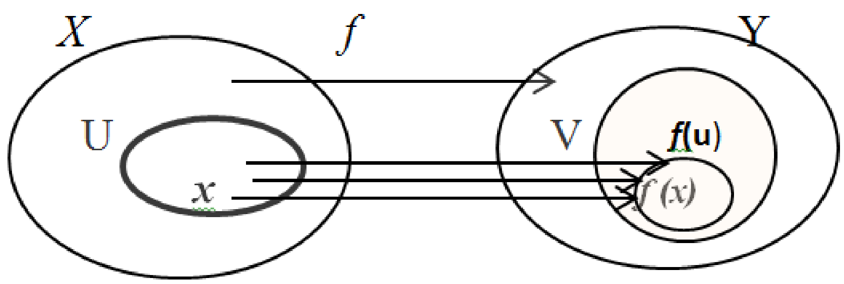
Proof:

If *f* is a closed, then $f(\overline{A})$ is a closed, since $f(A), \subset f(\overline{A})$ we abtain $\overline{f(A)} \subseteq \overline{f(\overline{A})} = f(\overline{A})$, as reqnired. Conversely, if the condition holds and A is a closed, then $f(A) \subset \overline{f(A)} \subset f(\overline{A}) = f(A)$, shows that

$\overline{f(A)} = f(A)$, so that $f(A)$ is a closed.■

Definition: 1-3-4 [16]

A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous at a point $x \in X$ iff for every open set V containing $f(x)$ there is an open set U containing x , such that $f(U) \subseteq V$.



We say that f is continuous on a set $A \subseteq X$ iff it is continuous at each point of A.

Definition: 1-3-5 [7]

Let (X, τ) and (Y, τ^*) be two spaces. A map $f : X \to Y$ is called continuous if the inverse image of each open set in Y is open set in X .

Example: 1-3-6

A constant map $f : X \to Y$ is always continuous . Since the inverse image of any open set U in Y, is either ∅ or X, and both are open.

Remark: 1-3-7 [7]

1) If $f : X \to Y$ and $g : Y \to Z$,are continuous, so also is $g \circ f : X \to Z$.

2) If f : X → Y continuous and A ⊂ X is taken with the subspace topology, then $f/_A$: A → Y is continuous, where $f/_A$ is the restriction of f on A .

3) If f : X → Y is continuous and f(X) is taken with the subspace topology , then f : X →f (X) is continuous .

Theorem: 1-3-8 [20]

If f : (X , τ_1) → (Y, τ_2) is a function , then the following statements are equivalent :

1- f^{-1} (C) is closed, where C is a closed in Y.

2- f^{-1} (U) ∈ τ_1, for every U ∈ τ_2.

3- f is continuous.

4- $f(\overline{A}) \subseteq \overline{f(A)}$, for every A ⊆ X.

Proof:

We demonstrate the equivalence by establishing the cycle of implications **(1)** → **(2)** → **(3)** → **(4)** → **(5)**.

(1) → **(2)**. Let U ∈ τ_2 ,then Y\U is a closed, which implies that f^{-1} (Y\U) is a closed. Since f^{-1} (Y\U) = f^{-1} (Y)\ f^{-1}(U) = X \ f^{-1}(U), we have that f^{-1}(U) ∈ τ_1.

(2) → **(3)**. Let x ∈ X and f (x) ∈ U∈ τ_2. Then x ∈ f^{-1} (U) ∈ τ_1, and since f (f^{-1}(U)) ⊂ U. f is continuous at x and thus continuous, since x was arbitrary.

(3) → (4). Let $A \subset X$. If $y \in f(\bar{A})$ and $y \in V \in \tau_2$, then $y = f(x)$ for some $x \in \bar{A}$. Since f is continuous, there exists an open set $U \in \tau_1$, such that $x \in U$ and $f(U) \subset V$. Also, $x \in \bar{A}$ implies that there is $p \in U \cap A$. Thus $f(p) \in f(U) \cap f(A) \subset V \cap f(A)$. Hence $y \in \overline{f(A)}$ and $f(\bar{A}) \subset \overline{f(A)}$.

(4)→(1). Let C be a closed subset of Y. Then $f(\overline{f^{-1}(C)}) \subset \overline{f(f^{-1}(C))} \subset \bar{C} = C$. This implies that $(\overline{f^{-1}(C)}) \subset f^{-1}(C)$ and so $f^{-1}(C)$ is a closed. ■

The previous theorem characterizes continuous functions as those having the property that the inverse images of open sets are open and the invers images of closed sets are closed. However, continuous function does not necessarily map open sets onto open set and closed sets onto closed sets, as the following example illustrates.

Example :1-3-9

Let X be the set of real numbers and $\tau = \{\emptyset\} \cup \{G \subset X : X \setminus G \text{ is countable}\}$, τ *is* a topology on X called (the Co - countable topology), let Y = [0 ,1] and $\tau_Y = \{G \cap [0,1] : G \in \tau\}$.

Then τ_Y is the subspace topology : induced on Y by τ.

Let $f : (R, \tau) \to (y, \tau_Y)$ defined by $f(x) = \begin{cases} x & if\ x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

Then f is not continuous, since $(0,1) \in \tau_Y$, but $f^{-1}[(0,1)] = (0,1) \notin \tau$, as $R \setminus (0,1)$ is uncountable.

Definition: 1-3-10 [20], [16]

A function h: (X, τ) $\rightarrow$ (Y, τ^*) is a homeomorphism (topological mapping) iff h is one-to-one, and onto, and h, h^{-1} are continuous.

Definition: 1-3-11 [20]

A property P of a topological space is topological property iff P is invariant (preserved) under homeomorphisms.

Remark: 1-3-12 [20]

The relation (X,τ) is homeomorphic to (Y,τ^*) is an equivalence relation on the collection of all topological spaces.

Definition: 1-3-13

A function f: (X, τ) $\rightarrow$ (Y, τ^*) is said to be an *embedding* if it is one-to-one, open, and continuous.

1-4 Product Spaces And Sequences In Spaces:

Definition:1-4-1

Let (X, τ) be a topological space. A family $\mathcal{B} \subset \tau$ is called a basis for τ if each open set (that is, member of τ) is the union of members of $\mathcal{B}$.

Definition: 1-4-2

Let (X_α, τ_α) be a topological space for all $\alpha \in \Gamma$. The topology defined on the set X = $\prod_{\alpha \in \Gamma} X_\alpha$ whose subbase the collection

S = $\{\pi^{-1}(U) : U \text{ is open in } X_\alpha,\ \alpha \in \Gamma\}$ is called the product topology or the Tychonoff topology on X .Where $\pi_\beta : \prod_{\alpha\in\Gamma} X_\alpha \to X_\beta$ defined by.

$\pi_\beta\left((X_g)_{\alpha\in\Gamma}\right) = x_\beta$ is β the projection map.

The Tychonoff topology has a base the collection

$$\beta = \{\bigcap_{i=1}^{n} \pi_{\alpha i}^{-1}(U_{\alpha i}) : U_{\alpha i} \text{ is open in } X_{\alpha i}, \alpha_i \in \Gamma, n \in N\}.$$

Theorem: 1-4-3

If $\prod_{\alpha\in\Lambda} X_\alpha$ is a product space, then each projection map is continuous and open.

Proof:

Let $\prod_{\alpha\in\Lambda} X_\alpha$ be a product space, where $\pi_\beta : \prod X_\beta \to X_\beta$ be the β- the projection map. If U is an open set in X_β, then $(\Pi_\beta^{-1} U)$ is a subbasic open set in $\prod_{\alpha\in\Lambda} X_\alpha$ for all $\alpha \in \Lambda$, so π_β is continuous, To show π_β is an open mapping. Let V is an open subset in $\prod_{\alpha\in\Lambda} X_\alpha$.If z $\in \pi_\beta(v)$. Then there exists x $\in$V such that $\pi_\beta(x) = z$ i-e $x_\beta = z$. Let B be a basic open set in $\prod_{\alpha\in7} X_\alpha$ such that x$\in B \subset V$, then $\pi_\beta(B)$ is open in X_β and z $\in \pi_\beta(B) \subset \pi_\beta(V)$. So $\pi_\beta(V)$ is open.

Definition: 1-4-4 [20]

Let (X,τ) be a topological space,$\{x_n\}_{n\in N}$ a sequence in X, and x $\in$ X. We shall say that $\{x_n\}_{n\in N}$ converges to x and write $x_n \to x$ iff $x \in G \in \tau$, implies that there is $n_0 \in N$ such that $x_n \in G$ for all $n \geq n_0$, where $N = \{1,2,3,\dots\}$. As a sequential limit, x is also designated by $\lim_{n\to\infty} x_n$ i − e $x = \lim_{n\to\infty} x_n$

Example: 1-4-5

Let $\mathbb{R}$ be the set of real numbers and let τ be the cofinite topology on $\mathbb{R}$, let $x_n = n$ for all $n \in N$. If $x \in \mathbb{R}$ and $x \in G \in \tau$, then $(\mathbb{R}\backslash G)$ is finite and $x_n \in G$, for all $n \geq n_0$, for $n_0 \in N$. Thus $x_n \to x$ for all $x \in \mathbb{R}$.

Theorem: 1-4-6 [20]

Let (X,τ) be a topological space, $A \subset X$ and $x \in X$. If $\{x_n\}_{n\in N}$ is a sequence in A, such that $x_n \to x$, then $x \in \bar{A}$.

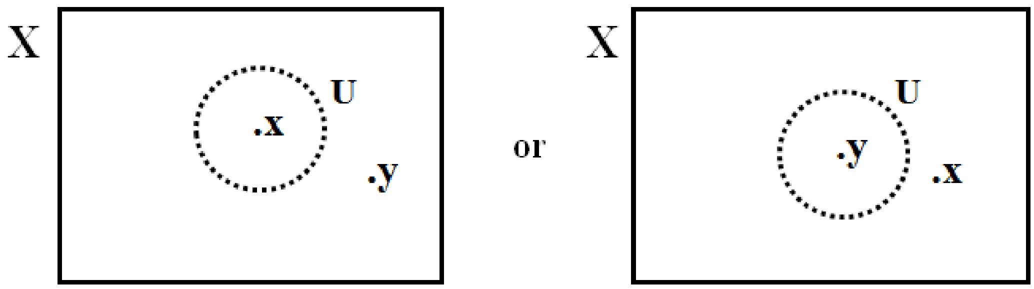
Proof:

Let $(x_n)_{n\in N}$ be a sequence and converging to x, and $x \in G \in \tau$. By Definition (1-4-4) there is $n_0 \in N$ such that $x_n \in G$, for all $n \geq n_0$. Since $x_n \in A$, for all $n \in N$, we have $G \cap A \neq \emptyset$, and $x \in \bar{A}$.■

1-5 Separation Axioms:

Definition: **1-5-1 [20]**

(X,τ) is a T_0- space iff $x, y \in X$ with $x \neq y$ implies that there exists $U \in \tau$ such that either $x \in U$ and $y \in X\backslash U$, or $y \in U$ and $x \in X\backslash U$



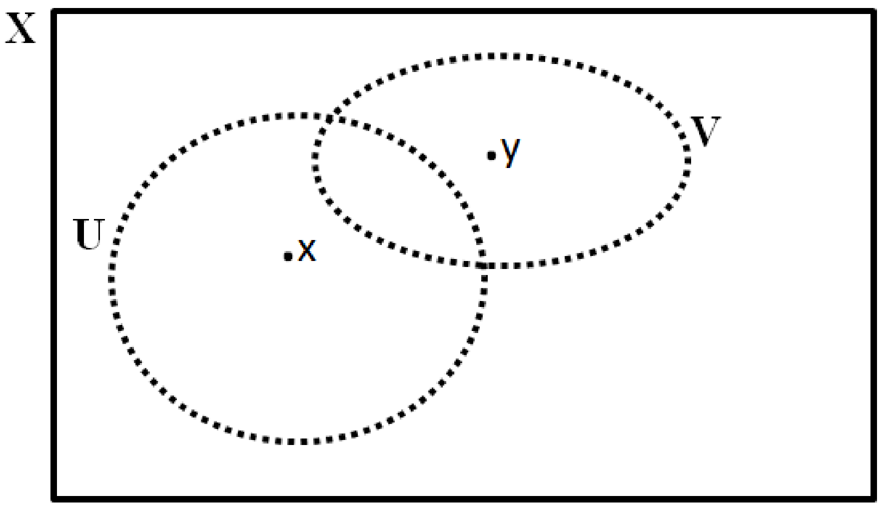
T_0- space T_0- space

Example/ 1-5-2

Let X = {a,b ,c} and $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X , τ_1) is a T_0-space . Let $\tau_2 = \{\emptyset, X\}$, then (X, τ_2) is not a T_0-space.

Definition: 1-5-3 [20]

(X, τ) is a T_1-space iff x, y ∈ X with x ≠ y implies that there exists U, V ∈ τ, with x ∈ U, y∈ X\U and y ∈ V, x ∈ X\V.



T_1-space

Theorem: **1-5-4 [20]**

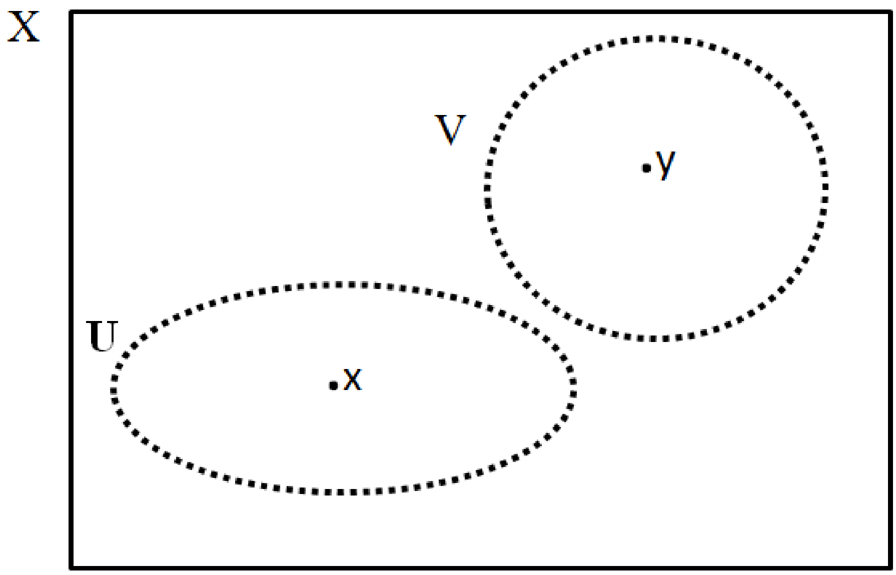
(X, τ) is a T_1-space iff $\overline{\{x\}} = \{x\}$, for each x ∈ X.

Proof:

Let (X, τ) be a T_1- space and $x \in X$. If $y \in X\backslash \{x\}$, then there exists $V\in \tau$ such that $y\in V$ and $x\in X\backslash V$. Hence $y\notin \overline{\{x\}}$ and $\overline{\{x\}}=\{x\}$. Conversely, suppose that $\overline{\{x\}} = \{x\}$, for each $x\in X$. Let $y, z \in X$ with $y \neq z$.Then $\overline{\{y\}} = \{y\}$ implies that there exists $V \in \tau$ such that $z \in V$, and $y \in X\backslash V$. Also, $\overline{\{z\}} = \{z\}$ implies that there exists $U \in \tau$ such that $y \in U$ and $z \in X\backslash U$. Thus (X, τ) is a T_1- space. ■

Definition: 1-5-5 [20] [16]

(X, τ) is a T_2 - space iff $x, y \in X$ with $x \neq y$, implies that there exists $U, V\in \tau$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$. T_2- spaces are also called Hausdorff spaces.



T_2 - space

Example: 1-5-6

Let X be any non-empty set, τ is the discrete topology on X, then (X, τ) is a T_2-space "Hausdorff space".

Clearly from the definitions that every T_2-space is a T_1-space and every T_1-space is a T_0-space. The following theoems holds with every T_i for $i = 0, 1, 2$.

Theorem: 1-5-7 [20]

If (X,τ) is a T_i-space, then every subspace of (X, τ) is also a T_i- space, for $i = 0,1,2$.

Proof:

We prove the theorem of the case i = 2, the other cases will be similar. Let $\emptyset \neq A \subset X$,and x, y $\in$ A with $x \neq y$ since x, y $\in$ X, there exists U, V $\in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus $x \in A \cap U$, $y \in A \cap V$, and we have $(A \cap U) \cap (A \cap V) = \emptyset$, since $U \cap V = \emptyset$. Hence (A, τ_A) is a T_2-space. ■

Theorem: 1-5-8 [20]

If $X = \prod_{\alpha \in \Lambda} X_\alpha$, then X is a T_i-space iff X_α is a T_i-space, for all $i = 0,1,2$.

Proof:

We will prove the theorem for i = 2, the other cases will be similar.

$\Longrightarrow$: If = $X = \prod_{\alpha\in\Lambda} X_\alpha$ is a T_2-space , then since for any $\alpha \in \Lambda$, X_α can be embedded as a subspace of $X = \prod_{\alpha\in\Lambda} X_\alpha$ and since every subspace of a T_2-space is a T_2-space , so X_α is a T_2-space for all $\alpha \in \Lambda$.

$\Longleftarrow$: Suppose X_α is a T_2-space for all $\alpha \in \Lambda$, let x, y $\in \prod_{\alpha\in\Lambda} X_\alpha$, $x \neq y$, there exists $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$. Since X is a T_2-space , there are disjoint open sets U , V in X_β containing x_β , y_β respectively, and hence $\prod_\beta^{-1}(U)$, $\prod_\beta^{-1}(V)$ are open sets in $\prod_{\alpha\in\Lambda} X_\alpha$ containing x and y respectively . So $\prod_{\alpha\in\Lambda} X_\alpha$ is a T_2-space. ■

Definition: 1-5-9 [7]

(X, τ) is a regular space if for each closed subset A of X, and x is a point of X not in A, then there exist two disjoint open sets one containing A and the other containing x.

Example: 1-5-10

Let X = {a,b,c} and $\tau = \{\emptyset, X\,\{b\}, \{a, c\}\}$, then (X ,τ) is a regular space.

Theorem :1-5-11 [20]

(X,τ) is a regular space iff $x \in U \in \tau$ implies that there exist $V \in \tau$ such that $x \in V \subset \overline{V} \subset U$.

Proof:

⟹: Let (X, τ) be a regular and $x \in U \in \tau$,then $X\backslash U$ is closed and there exists $G_1, G_2 \in \tau$ with $x \in G_1$, and $X\backslash U \subset G_2$ and $G_1 \cap G_2 = \emptyset$. Thus $x \in G_1 \subset X\backslash G_2 \subset U$. Also, $\overline{G_1} \subset X\backslash G_2$ since $X\backslash G_2$ is a closed. Let V= G_1 . Now assume that $x \in U \in \tau$ implies there exists $V \in \tau$ such that $x \in V \subset \overline{V} \subset U$. Let $x \in X$ and $F \subset X\backslash\{x\}$, with F closed. Then $x \in X\backslash F \in \tau$. Hence there exists $V \in \tau$ such that $x \in V \subset \overline{V} \subset X\backslash F$.

This implies that $F \subset X\backslash\overline{V} \in \tau$. And $(X\backslash \overline{V}) \cap V = \emptyset$. Thus (X , τ) is a regular space.■

Theorem: 1-5-12 [7]

Every subspace of a regular space is a regular space.

Proof:

Given $Y \subset X$, let $B \subset Y$ be closed in Y, and $x_0 \in Y\backslash B$.Then $B = Y \cap A$, where A is a closed in X, and since A does not contain x_0 , there are disjoint open sets U, V in X , such that $x_0 \in U$, $A \subset V$.

Then $U \cap Y$ and $V \cap Y$ are the required disjoint open sets of Y respectively containing x_0 and B respectively. ■

Theorem: **1-5-13 [7]**

If $X = \prod_{\alpha \in \Lambda} X_\alpha$, then X is a regular space iff X_α is a regular for all $\alpha \in \Lambda$.

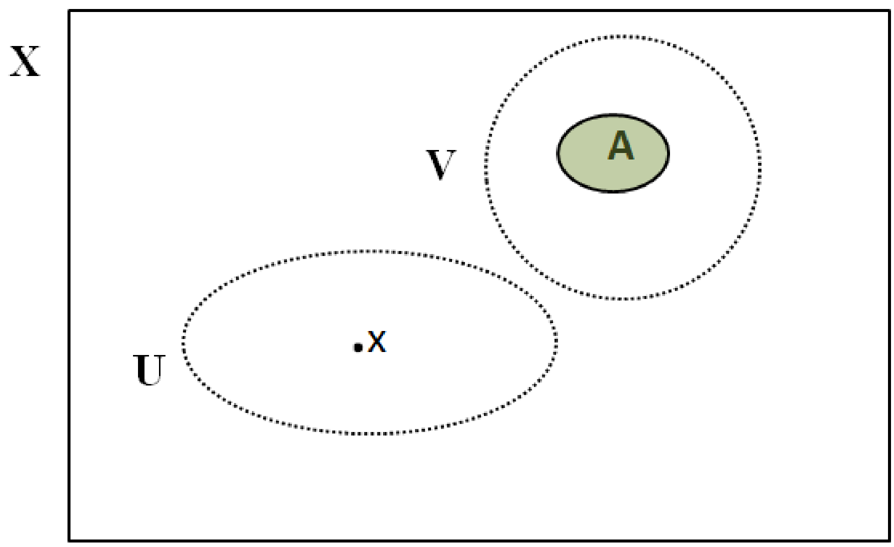
Proof:

⟹: If $X = \prod_{\alpha\in\Lambda} X_\alpha$ is regular, then implies that each X_α is regular (A subspace of regular space is regular),since each X_α is homeomorphic to a subspace of X .

⟸: Conversely, suppose each X_α is regular and that $U = \prod_{\alpha\in\Lambda} U_\alpha$ is a basic open set containing x. For each α we can pick an open set V_α in X_α such that $x_\alpha \in V_{\alpha i} \subset \overline{V}_\alpha \subset U_\alpha$. Let $V = \prod_{\alpha\in\Lambda} V_\alpha$ then V is an open set in $\prod_{\alpha\in\Lambda} X_\alpha$ and $x \in V \subset \overline{V} \subset U$. Therefore $\prod_{\alpha\in\Lambda} X_\alpha$ is regular .■

Definition: 1-5-14 [20]

(X, τ) is called a T_3-space iff (X, τ) is a regular and T_1-space.



T_3-space

Theorem: 1-5-15

Every subspace of a T_3-space is a T_3-space.

Theorem: 1-5-16

$X = \prod_{\alpha\in\Lambda} X_\alpha$ is a T_3-space iff X_α is a T_3-space for all $\alpha \in \Lambda$.

Definition: **1-5-17 [20]**

(X,τ) is a completely regular iff $x \in X$ and $A \subset X\backslash\{x\}$, with A is closed, implies the existence of a continuous function $f : X \rightarrow I$ with $f(x) = 0$ and $f(A) = \{1\}$, I is the unit interval i-e $I = [0,1]$

Definition: 1-5-18 [20]

(X, τ) is a Tychonoff ($T_{3\frac{1}{2}}$) space iff (X, τ) is completelty regular,

T_1-space.

Theorem: 1-5-19 [7]

Every subspace of completely regular space is completely regular.

Proof:

Let $Y \subset X$ be a subspace , and let $x \in Y$, A is a closed in Y. since $A = Y \cap F$, where F is closed in X , $x \notin A$, then $x \notin Y \cap F$, implies $x \notin F$,since X is completely regular there is a continuous function $f : X \rightarrow I$, such that $f(x) = 0$, $f(F) = 1$ .Let $g = f/_Y$ then $g : Y \rightarrow I$ is continuous and since $A \subset F$, then $g(x) = 0$, $g(A) = 1$, thus Y is a completely regular space .■

Theorem: 1-5-20

Every subspace of a Tychonoff space is a Tychonoff ($T_{3\frac{1}{2}}$) space.

Theorem: 1-5-21

If $X = \prod_{\alpha\in\Lambda} X_\alpha$, then X is a completely regular iff X_α is a completely regular for all $\alpha \in \Lambda$.

Proof:

⟹: Suppose $X = \prod_{\alpha\in\Lambda} X_\alpha$ is completely regular space, since each X_α can be embedded as a subspace of $X = \prod_{\alpha\in\Lambda} X_\alpha$.

So X_α is a completely regular space for all $\alpha \in \Lambda$.

⟸: Suppose X_α is a completely regular for all $\alpha \in \Lambda$. Let A be a closed set in $\prod_{\alpha\in\Lambda} X_\alpha$, $x \notin A$ there exist a basic nbhd $\bigcap_{i=1}^{n} \pi_{\alpha i}^{-1}(U_i)$ containing x and $[\bigcap_{i=1}^{n} \pi_{\alpha i}^{-1}(U_i)] \cap A = \emptyset$, so $x_{\alpha_k} \notin X \setminus U_k$ for all k , there exist a continuous function , $f_k : X_{\alpha_k} \to I$ such that $f_k(x_{\alpha_k}) = 0$, $f_k(X_{\alpha_k} \setminus U_k) = 1$.

But then $f_k \circ \pi_{\alpha_k}$ is continuous function, $f_k \circ \pi_{\alpha_k} = \prod_{\alpha\in\Lambda} X_\alpha \to I$ for all k .

Let $f : \prod_{\alpha\in\Lambda} X_\alpha \to I$ defined by. $f(y) = \max \{(f_k \circ \pi_{\alpha_k})(y)\}_{k=1}^{n}$ .Then f is continuous, $f(x) = 0$, $f(A) = 1$ .So $\prod_{\alpha\in\Lambda} X_\alpha$ is completely regular .■

Theorem: 1-5-22

$X = \prod_{\alpha\in\Lambda} X_\alpha$ is a Tychonoff space iff X_α is a Tychonoff space for all $\alpha \in \Lambda$.

Definition: 1-5- 23 [20]

(X, τ) is a normal space iff for every pair F_1, F_2 of disjoint closed subsets of X, there exists $G_1, G_2 \in \tau$ with $F_1 \subset G_1$, $F_2 \subset G_2$,and $G_1 \cap G_2 = \emptyset$.

Example: 1-5-24

Let $X = \{u, v, z, w\}$ and $\tau = \{\emptyset , \{u\},\{u , v\} ,\{u , v , w \} , X\}$.

Note that closed sets are $\emptyset$, $\{z\}, \{w, z\}, \{v, w, z\}, X$, every non-empty closed set contains z , so (X, τ) is a normal space.

Theorem: 1-5-25 [7]

A topological space X is normal iff for any closed set F and open set U containing F, there exists an open set V such that $F \subseteq V \subseteq \overline{V} \subseteq U$.

Proof:

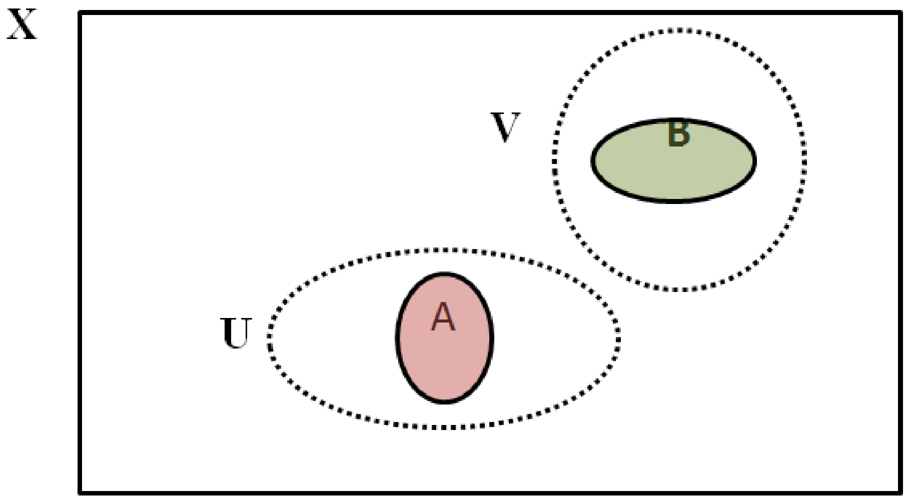
$\Longrightarrow$: Suppose X is normal and the closed set F is contained in the open set U. Then k = X\U is a closed set which is disjoint from F. By normality, there exist two disjoint open sets G and H such that $F \subseteq G$ and $k \subseteq H$. Since $G \subseteq X\backslash H$ we have $\overline{G} \subseteq (\overline{X\backslash H}) = (X\backslash H) \subseteq X\backslash K = U$. Thus G is the desired set.

$\Longleftarrow$: Now suppose the condition holds, and let F_1 and F_2 be disjoint closed subsets of X. Then F_1 is contained in the open set $X\backslash F_2$ and by hypothesis, there exists an open set V such that $F_1 \subseteq V \subseteq \overline{V} \subseteq X\backslash F_2$.

Clearly, V and $X\backslash\overline{V}$ are the two desired disjoint open sets containing F_1 and F_2 , respectively. ■

Definition: 1-5-26 [20]

(X,τ) is a T_4-space iff (X,τ) is a normal T_1-space.



T_4-space

Theorem: 1-5-27

If X is a non-empty set, τ_1, τ_2 are topologies on X, with $\tau_1 \subset \tau_2$. Then:

1) If (X,τ_1) is a T_i-space, then (X,τ_2) is a T_i-space for $i = 0,1,2$.
2) If (X,τ_1) is a regular space, then (X,τ_2) need not to be a regular space.
3) If (X,τ_1) is a normal space, then (X,τ_2) need not to be a normal space.

Proof:

1) We prove the theorem for the case i = 2, the other cases are similar. Let (X,τ_1) be a T_2-space $x \neq y$ in X, there exists two disjoint open sets U, V in τ_1 with $x \in U$, $y \in V$, since $\tau_1 \subset \tau_2$. So every open set in τ_1 is an open set in τ_2. Therefore for every pair of distinct points x, y in X, there exists two disjoint open sets U, V in τ_2, such that $x \in U$, and $y \in V$. Thus (X,τ_2) is a T_2-space.
2) For example, let $X=\{a, b, c\}$, $\tau_1=\{\emptyset, X \{b\}, \{a, c\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Note that $\tau_1 \subset \tau_2$ but (X , τ_1) is a regular space , and τ_2 is not a regular space , since closed sets in (X ,τ_2) are $\emptyset$,X,{b},{c},{b,c}{a,c}, there is no open set contain {b,c} , but not a .
3) For example , let $X=\{a, b, c\}$,and $\tau_1=\{\emptyset, X\}$, $\tau_2 =\{\emptyset, X, \{a,b\}$,{b}, {b,c}} , note that (X ,τ_1) is a normal space but (X ,τ_2) is not a normal space .■

Definition: 1-5-28 [20]

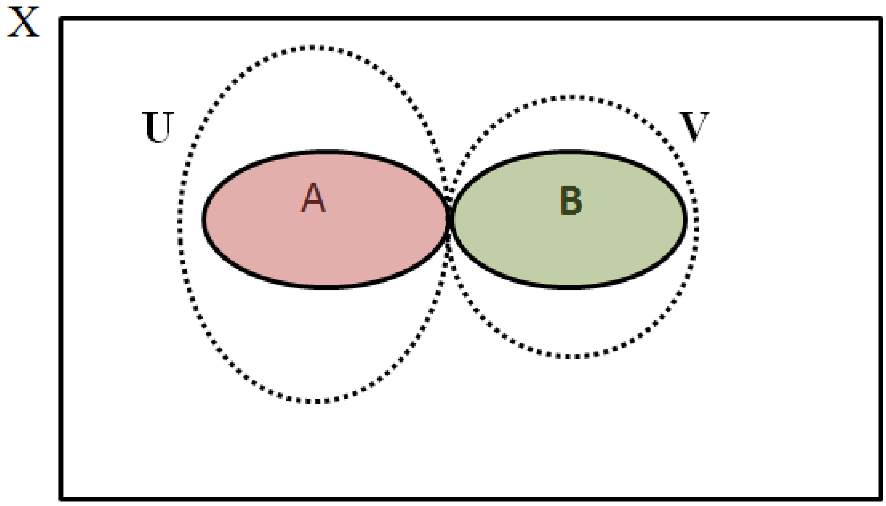
Let (X,τ) be a topological space and A, B be subsets of X, A and B are separated iff $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Definition: 1-5-29 [20]

(X,τ) is completely normal iff for every pair A, B of separated subsets of X, there are disjoint open sets U and V with $A \subset U$, and $B \subset V$.

Definition: 1-5-30 [20]

(X,τ) is a T_5-space iff (X,τ) is a completely normal T_1-space.



T_5-space

Theorem: 1-5-31 [20]

(X,τ) is completely normal iff it is hereditarily normal i-e, each subspace of (X,τ) is a normal.

Proof:

⟹: Let (X,τ) be a completely normal space and (A, τ_A) be any subspace of (X,τ). If C_1 and C_2 are disjoint subsets of A that are closed in A , then there are closed subsets F_1 and F_2 of X such that $C_1 = A \cap F_1$

and $C_2 = A \cap F_2$, thus $\bar{C}_1 \subset F_1$ and $\bar{C}_2 \subset F_2$. This implies that $\bar{C}_1 \cap C_2 = C_1 \cap \bar{C}_2 = \emptyset$, and C_1 , C_2 are separated in X. the complete normality of (X, τ) implies that there exist $G_1, G_2 \in \tau$ such that $C_1 \subset G_1$, $C_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. Finaly, $C_1 \subset A \cap G_1 \in \tau_A$ and $C_2 \subset A \cap G_2 \in \tau_A$ with $(A \cap G_1) \cap (A \cap G_2) = \emptyset$. This estabishes the normality of (A, τ_A).

$\Leftarrow$: Conversely, let each subspace of (X, τ) be normal, and let C_1 and C_2 be separated in X. If $A = X \setminus [(\bar{C}_1 - C_1) \cup (\bar{C}_2 - C_2)]$ has the relative topology τ_A , then C_1 and C_2 are closed in A .The normality of (A, τ_A) implies there exist $A \cap G_1$, $A \cap G_2 \in \tau_A$ such that $C_1 \subset A \cap G_1$, $C_2 \subset A \cap G_2$ and $(A \cap G_1) \cap (A \cap G_2) = \emptyset$

This implies that $(G_1 \cap G_2) \subset (\overline{C_1} - C_1) \cup (\overline{C_2} - C_2)$.

Let $U_1 = G_1 \cap (X - \overline{C_2}) \in \tau$ and $U_2 = G_2 \cap (X - \overline{C_1}) \in \tau$.

Clearly $C_1 \subset U_1$ and $C_2 \subset U_2$ with $U_1 \cap U_2 = \emptyset$.

This establishes the complete normality of (X, τ). ■

Chapter two

Fibrewise topological spaces and separation axioms in fibrewise topological spaces

The purpose of this chapter is to introduce the concept of fibrewise topological spaces, to study consider separation axioms in fibrewise topological spaces, fibrewise T_0-spaces, fibrewise T_1-spaces, fibrewise T_2- (Hausdorff) spaces, fibrewise regular spaces, fibrewise completely regular, fibrewise normal and fibrewise completely normal spaces, and to give several results which are needed in the chapter of separation axioms in fibrewise ideal topological spaces.

2-1 Definitions and examples;

Definition :2-1-1 [2],[15]

Let B be any non-empty set. Then a fibrewise set over B consists of a set X together with function p: X $\rightarrow$ B. called the projection, and B is called the base set. For each point b of B, the fibres over b is the subset $X_b = p^{-1}(b)$ of X. Fibres may be empty since we do not require p to be surjective , also for each subset B^* of B, we regard $X_{B^*} = p^{-1}(B^*) = \bigcup_{b \in B^*} X_b$ is a fibrewise set over B^* with the projection determined by p .

Remark: 2-1-2 [2]

Let X be a fibrewise set over B, with projection p. Then Y is a fibrewise set over B, with projection p o q for each set Y and function q: Y $\rightarrow$ X.

Definition: 2-1-3 [2],[15]

If X and Y are fibrewise sets over B, with projections p and q respectively, a function Ψ: X $\rightarrow$ Y is said to be fibrewise function if q Ψ = p, in other words if $\Psi(X_b) \subseteq Y_b$ for each b $\in$ B.

Definition: 2-1-4 [2],[1]

Let B be a topological space, then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

Example: 2-1-5

Let X = { x ,y , z } , B = { a , b , c} and P : X $\rightarrow$ B be a map defined by P(x) = a , P (y) = b , and P(z) = c , let $\tau_1 = \{X, \phi, \{x\}, \{y\}, \{x, y\}\}$ be a topology on X , and $\tau_2 = \{B, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a topology on B, then P : X $\rightarrow$B is a continuous function . Therefore (X, τ) is fibrewise topological space over B.

Example: 2-1-6

If X =$\mathbb{R}$ with τ the usual topology and B = $\mathbb{R}$ with the usual topology , P : X $\rightarrow$ B defined by P(x) = |x| , then (X ,τ) is fibrewise topological space over B , and for any b $\in$ $\mathbb{R}$;

$$X_b = P^{-1}(b) = \begin{cases} \{-b, b\} & \text{if } b > 0 \\ \emptyset & \text{if } b < 0 \\ \{0\} & \text{if } b = 0 \end{cases}$$

Remark: 2-1-7 [15]

The coarsest such topology is the topology induced by p, in which the open sets of X are precisely the inverse images of the open sets of B; this is called the fibrewise indiscrete topology.

Example: 2-1-8

If X is any discrete topological space and B is any topological space,and $P : X \rightarrow B$ is any map , then X is a fibrewise topological space over B.

Definition: 2-1-9

Let (X_α, τ_α) be a fibrewise topological space over B for all $\alpha \in \Lambda$, then the product $X = \prod_{\alpha\in\Lambda} X_\alpha$ defines fibrewise spaces over B and equipped with the family of fibrewise projection

$P_\alpha o \pi_\alpha: \prod_{\alpha\in\Lambda} X_\alpha \rightarrow B$. Where $\pi_\alpha: \prod_{\alpha\in\Lambda} X_\alpha \rightarrow X_\alpha$ is the α - the projection map and $P_\alpha: X_\alpha \rightarrow B$ is the projection of X_α.

Theorem: 2- 1-10

If (X, τ_1) is a fibrewise topological space over B, and τ_2 is a topology on X such that $\tau_1 \subset \tau_2$. Then (X, τ_2) is a fibrewise topological space over B.

Proof:

Since every open set in τ_1 is in τ_2, and since (X, τ_1) is fibrewise topological space over B, then the projection p: (X, τ_1) $\to$ B is continuous. So p: (X, τ_2) $\to$ B is continuous also. Therefore (X, τ_2) is fibrewise topological space over B. ■

Definition: 2-1-11 [15]

A fibrewise function Ψ: X $\to$ Y, where X and Y are fibrewise topological spaces over B is called:

a) Continuous if for each point $x \in X_b$ where $b \in B$, the inverse image of each open set containing Ψ(x) is an open set of X containing x.

b) Open if for each point $x \in X_b$ where $b \in B$, the direct image of each open set containing x is an open set containing Ψ (x).

Propositions: 2- 1-12 [1]

Let φ: X $\to$ Y be a fibrewise function, where Y is fibrewise topological space over B, and X is a fibrewise set has the induced fibrewise topology. Then for each fibrewise topological space Z, a fibrewise function ψ : Z $\to$ X is continuous iff the composition $\varphi \circ \psi : Z \to Y$ is continuous .

Proof:

⟹: Suppose that ψ is continuous. Let $z \in Z_b$, where $b \in B$, and V open set containing $(\varphi o \psi)(z) = y_b$ in Y. Since φ is continuous, $\varphi^{-1}(V)$ is an open set containing $\psi(z) = x \in X_b$ in X . Since ψ is continuous , then $\psi^{-1}(\varphi^{-1}(V))$ is an open set containing $z \in Z_b$ in Z , and $\psi^{-1}(\varphi^{-1}(V)) = (\varphi o \psi)^{-1}(V)$ is an open set containing $z \in Z_b$ in Z .

⟸: Suppose that $\varphi o \psi$ is continuous. Let $z \in Z_b$, where $b \in B$ and U open set containing $\psi(z) = x \in X_b$ in X. Since φ is fibrewise function implies that φ is continuous so φ is open, $\varphi(U)$ is an open set containing $\varphi(x) = \varphi(\psi(z)) = y \in Y_b$ in Y .

Since $\varphi o \psi$ is continuous. Then $(\varphi o \psi)^{-1}(\varphi(U)) = \psi^{-1}(U)$ is an open set containing $z \in Z_b$ in Z. ■

Definition: 2-1-13

The fibrewise topological space X over B is called fibrewise closed (resp.open) if the projection p is closed (resp. open) function.

Propositions: 2-1-14

Let φ : X →Y be an open fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise open, then X is fibrewise open.

Proof:

Suppose that φ : X→Y is fibrewise open function and Y is fibrewise open, the projection p_Y: Y→ B is open. To show that X is fibrewise open, we need to show the projection p_X: X → B is open. Now if V is open subset of X_b , where b ∈ B , since φ is open , then φ(V) is open subset of Y_b , since p_Y is open , then p_Y (φ(V)) is open in B , but p_Y (φ(V)) = (p_Yoφ)(V) = p_X(V) is open in B .Thus p_X is open and X is fibrewise open. ■

Propositions: 2- 1-15

Let φ: X →Y be continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise closed then Y is fibrewise closed.

Proof:

Suppose that φ: X →Y is continuous fibrewise surjection and X is fibrewise closed, the projection p_X : X → B is closed . To show that Y is fibrewise closed, we need to show the projection p_Y:Y→B is closed . Let G be a closed subset of Y_b, where b ∈ B. Since φ is continuous fibrewise , then φ^{-1}(G) is closed subset of X_b . Since p_X is closed, then $p_X(\varphi^{-1}(G)) = (p_X o\ \varphi^{-1})(G) = p_Y(G)$ is closed in B. Thus p_Y is closed and Y is fibrewise closed. ■

Theorem: 2-1-16

If (X_α, τ_α) is a fibrewise topological spaces over B , for all $\alpha \in \Lambda$.

$P_\alpha : X_\alpha \to B$ is a projection map for all $\alpha \in \Lambda$.

Then for any $\beta \in \Lambda$, $\Pi_{\alpha\in\Lambda} X_\alpha$ is a fibrewise topological space over B with projection $P_\beta \circ \pi_\beta$.

Proof:

That is clear, since the projection map $P_\beta \circ \pi_\beta$ is continuous for all $\beta \in \Lambda$.■

Corollary: 2- 1-17

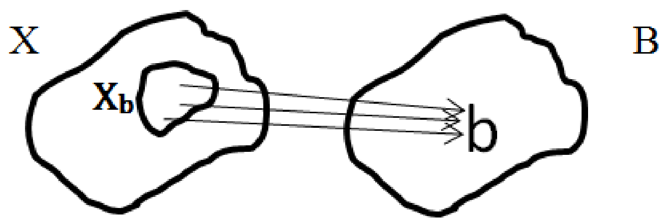
The product of fibrewise topological spaces over a topological space B is a fibrewise topological space over B.

2-2 Separation axioms in fibrewise topological spaces:

Before we introduce the definition of fibrewise separation axioms we introduce the following definitions.

Definition: 2-2-1

Let (X,τ) be a fibrewise topological space over B, $X_b \subset X$, $b \in B$ is called trivial fibre subspace if $X_b = p^{-1}(b) = \emptyset$ or one point set.



X_b is the fibre space corresponding to b.

Example: 2-2-2

If P: (ℝ, τ) → (ℝ, τ) is defined by $P(x) = x^2$ and τ is the usual topology on ℝ.

Then $X_b = p^{-1}(b) = \begin{cases} \{-\sqrt{b}, \sqrt{b}\} & \text{if } b > 0 \\ \{0\} & \text{if } b = 0 \\ \emptyset & \text{if } b < 0 \end{cases}$

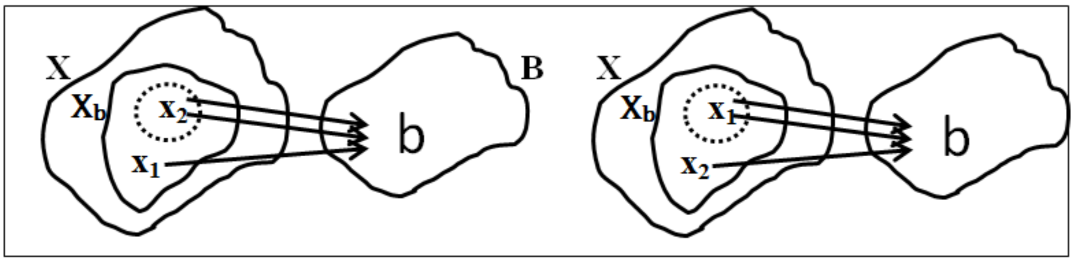
Then ∅ and {0} are trivial fibre subspace over B.

Theorem: 2-2-3

If (X,τ) is a fibrewise space over B. and p: X → B. the projection map is injective. Then every fibre subspace is trivial i- e X_b is empty or one- point set.

Definition: 2-2-4

A fibrewise topological space (X, τ) over B is said to be fibrewise T_0-space if every non-trivial fibre subspace X_b is T_0 -space, where b ∈ B. that is $x_1, x_2 \in X_b$ where b ∈ B and $x_1 \neq x_2$, either there exists an open set in X_b containing x_1 and does not contain x_2 in X or vice versa.



T_0 - space

Example: 2-2-5

Let $X = B = \mathbb{R}$ with the usual topology τ and let i be the identity projection function the $(\mathbb{R}, \tau)$ is fibrewise T_0- space over $(\mathbb{R}, \tau)$.

Here in this example $X_b = p^{-1}\{b\} = \{b\}$ is a trivial subspace for all $b \in \mathbb{R}$.

Theorem: 2-2-6

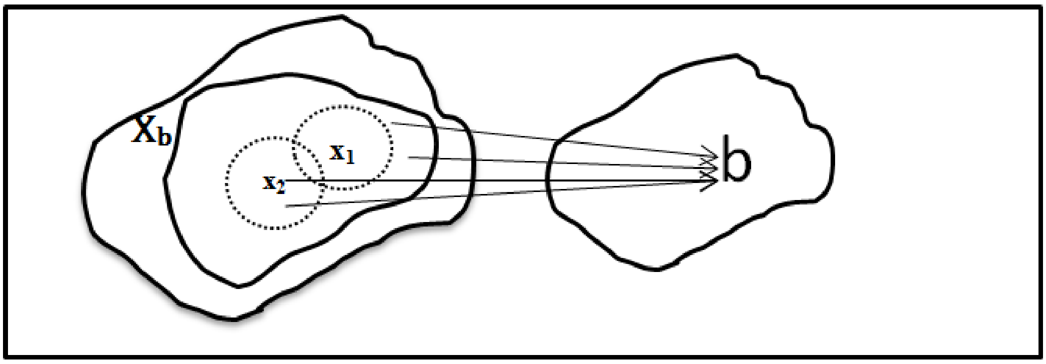
If (X,τ) is a T_0-space and is fibrewise space over BThen (X,τ) is fibrewise T_0-space over B.

Proof:

For any $b \in B$, X_b is fibre subspace of X, and since every subspace of a T_0-space is a T_0-space, so X_b is a T_0-space. ■

Definition: 2-2-7

A fibrewise topological space (X, τ) over B is said to be a fibrewise T_1- space if every non - trivial fibre subspace X_b is T_1- space .That is if $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist open sets U_1, U_2 in X_b , such that $x_1 \in U_1$, $x_2 \notin U_1$ and $x_2 \in U_2$, $x_1 \notin U_2$.



Fibrewise T_1- space

Example: 2-2-8

Let X = B = {a, b}. Let τ_1 = {X, ϕ, {a}, {b}(the discrete topology on X), τ_2 = {X, ϕ }(the trivial topology on B).

p: (X,τ_1) $\rightarrow$ (B, τ_2) defined by p(x) = x, then p is continuous, and for any z $\in$ B, $X_z = p^{-1}(z)$ = {z} is a trivial fibre subspace. So (X, τ) is a fibrewise T_1-space over B.

Theorem: 2-2-9

If (X,τ) is a T_1-space and is a fibrewise space over B. Then (X,τ) is a fibrewise T_1-space over B.

Proof:

For any b $\in$ B , X_b is a fibre subspace of X, and since every subspace of a T_1- space is a T_1-space. So X_b is a T_1-space. ■

Theorem: 2-2-10

Let (X, τ) be a fibrewise topological space over B. Then (X, τ) is a fibrewise T_1- space iff $\overline{\{x\}} = \{x\}$ for every $x \in X_b$.

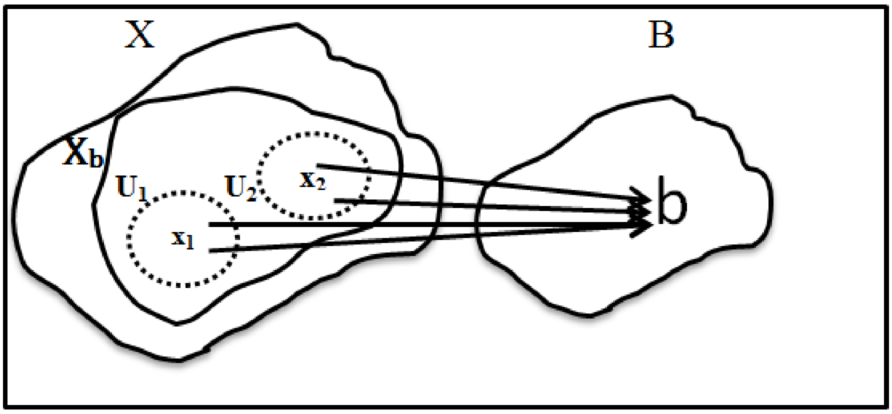
Proof:

⟹: Let (X,τ) be a fibrewise T_1- space over B and $x \in X_b$ if $y \neq x$, $y \in X_b \setminus \{x\}$, then there exists an open set $V \in \tau$, such that $y \in V$, and $x \notin V$. Hence $y \notin \overline{\{x\}}$ and $\overline{\{x\}} = \{x\}$.

⟸: Conversely, suppose that $\overline{\{x\}} = \{x\}$ for each $x \in X_b$. Let $y, z \in X_b$ with $y \neq z$. Then $\overline{\{y\}} = \{y\}$ implies that there exists $V \in \tau$ such that $z \in V$ and $y \notin V$. Also, $\overline{\{z\}} = \{z\}$ implies that there exists $U \in \tau$ such that $y \in U$ and $z \notin U$. Hence (X,τ) is a fibrewise T_1- space over B by (definition 2-2-8). ■

Definition: 2-2-11

A fibrewise topological space X over B is called a fibrewise T_2(Hausdorff)-space if every non-trivial fibre subspace is T_2 (Hausdorff). That is if $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist disjoint open sets U_1, U_2 containing x_1, x_2 in X_b.



Fibrewise T_2- space

Example / 2-2-12

If X = ℝ,τ the cofinite topology and let B = ℝ with the cofinite topology and P : X → B is defined by $P(x) = x^2$ then p is continuous . So (X ,τ) is fibrewise topological space over B , and for any b ∈ B

$$X_b = \begin{cases} \{-\sqrt{b}, \sqrt{b}\} & \text{if } b > 0 \\ \{0\} & \text{if } b = 0 \\ \emptyset & \text{if } b < 0 \end{cases}$$

If b > 0, $X_b = \{-\sqrt{b}, \sqrt{b}\}$ is non-trivial subspace and the subspace is discrete subspace of X , then (X ,τ) is a fibrewise T_2-space , but not T_2-space .

Theorem: 2-2-13

If (X,τ) is a T_2-space and is a fibrewise space over B .Then (X,τ) is a fibrewise T_2- space over B .

Proof:

For any $b \in B$, X_b is a fibre subspace of X , and since every subspace of a T_2- space is a T_2- space , so X_b is a T_2- space . ■

Proposition: 2-2-14

Let $\varphi : X \to X^*$ be an embedding fibrewise function , where X and X^* are fibrewise topological spaces over B .If X^* is fibrewise Hausdorff . So is X.

Proof:

Let $x_1, x_2 \in X_b$, where $b \in B$, and $x_1 \neq x_2$.Then $\varphi(x_1), \varphi(x_2) \in X^*_b$ are distincet. Since X^* is fibrewise Hausdorff, there exist disjoint open sets V_1, V_2 containing $\varphi(x_1)$, $\varphi(x_2)$ in X^*_b, there inverse images $\varphi^{-1}(V_1)$, $\varphi^{-1}(V_2)$ are disjoint open sets containing x_1 , x_2 in X_b , and so X is fibrewise Hausdorff space . ■

Theorem: 2-2-15

If X_α is a fibrewise T_2- topological spaces over B ,with projection P_α for all $\alpha \in \Lambda$,then for all $\beta \in \Lambda$, $\Pi_{\alpha\in\Lambda} X_\alpha$ is a fibrewise T_2- topological space over B with projection $P_\beta \circ \pi_\beta$.

Proof:

Suppose X_α is a fibrewise T_2- space over B, for all $\alpha \in \Lambda$ and with projection $P_\alpha : X_\alpha \to B$ then for any $\beta \in \Lambda$, $P_\beta \circ \pi_\beta : \Pi_{\alpha\in\Lambda} X_\alpha \to B$ is continuous . So $\prod_{\alpha\in\Lambda} X_\alpha$ is a fibrewise topological space over B with projection $P_\beta \circ \pi_\beta$. Now for any $b \in B$, $X_b = (P_\beta \circ \pi_\beta)^{-1}(b) = \pi^{-1}_\beta(p^{-1}_\beta(b))$ $= \{ (x_\alpha)_{\alpha\in\Lambda} : x_\alpha \in X_\alpha,\ x_\beta \in p^{-1}(b) \}$ is a subspace of $\prod_{\alpha\in\Lambda} X_\alpha$ and since the product of T_2-spaces is a T_2-space, and every subspace of T_2- space is a T_2-space , so it follows that every fibre subspace is a T_2- space , and hence $\prod_{\alpha\in\Lambda} X_\alpha$ is fibrewise T_2- space over B . ■

We now proceed to consider the fibrewise versions of higher Separation Axioms starting with regularity.

Definition: 2-2-16

A fibrewise topological space X over B is called a fibrewise regular space if each non-trivial fibre subspace is regular, that is for each point $x \in X_b$, where $b \in B$ and for each open set Vof x in X_b , there exists neighborhood W of x in X_b such that $x \in W \subset \overline{W} \cap X_b \subset V$.

Remark: 2-2-17

If X is fibrewise regular space over B, then X^*_B is fibrewise regular space over B^* for each subspace B^* of B .

Theorem: 2-2-18

If (X,τ) is a regular space and a fibrewise topological space over B,

Then (X,τ) is a fibrewise regular space over B .

Proof:

Since every subspace of regular space is regular, so it follows that every fibre subspace is regular and hence (X,τ) is fibrewise regular space. ■

Proposition: 2-2-19

Let $\varphi : X \to X^*$ be a fibrewise embedding function , where X and X^* are a fibrewise topological spaces over B . If X^* is fibrewise regular, then so is X.

Proof:

Let $x \in X_b$ where $b \in B$ and let V be an open set containing x in X_b. Then $V = \emptyset^{-1}(V^*)$ where V^* is an open set containing $x^* = \emptyset(x)$ in X^*_b since X^* is fibrewise regular there exist an open set U^* containing x^* in X^*_b , such that $X^*_b \cap cl(U^*) \subset V^*$. Then $U = \emptyset^{-1}(U^*)$ is an open set containing x in X_b such tha $X_b \cap cl(U) \subset V$, and so X is fibrewise regular.■

The class of fibrewise regular spaces is fibrewise multiplicative in the following sense .

Proposition: 2-2-20

If X_α is a fibrewise regular topological space over B , with projection P_α for $\alpha \in \Lambda$. Then $X=\prod_{\alpha \in \Lambda} X_\alpha$ is fibrewise regular topological space over B , with projection $P_\beta \circ \pi_\beta$ for any $\beta \in \Lambda$.

Proof:

Suppose X_α is a fibrewise regular space over B, for all $\alpha \in \Lambda$ and with projection $P_\alpha : X_\alpha \to B$. Then for any $\beta \in \Lambda$, $P_\beta \circ \pi_\beta = \Pi_{\alpha\in\Lambda} X_\alpha \to B$ is continuous , so $\Pi_{\alpha\in\Lambda} X_\alpha$ is a fibrewise topological space over B , with projection $P_\beta \circ \pi_\beta$. Since for any $b \in B$, $X_b = (P_\beta \circ \pi_\beta)^{-1}(b) = \pi_\beta^{-1}(p^{-1}{}_\beta(b)) = \{(x_\alpha)_{\alpha\in\Lambda} : x_\alpha \in X_\alpha ,\ x_\beta \in p^{-1}(b)\}$ is a subspace of $\prod_{\alpha\in\Lambda} X_\alpha$ and since the product of regular spaces is a regular space , and every subspace of regular is a regular, then $\prod_{\alpha\in\Lambda} X_\alpha$ is a fibrewise regular space over B . ■

Proposition: 2-2-21

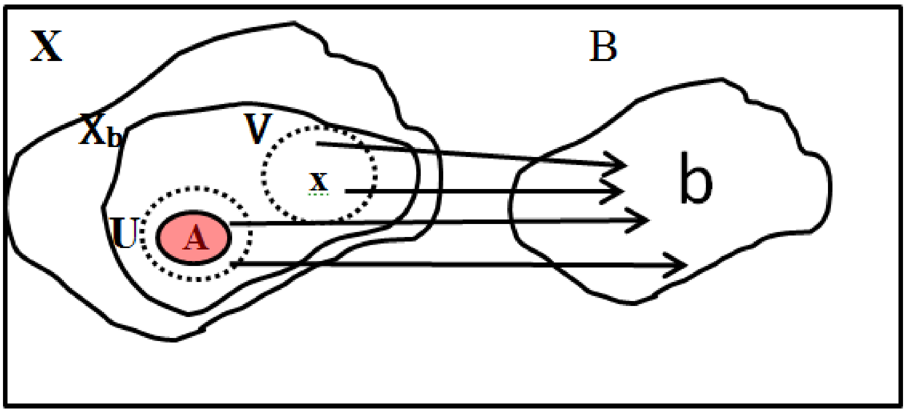
Let $\varphi : X \to Y$ be an open, closed and continuous fibrewise surjection, where X and Yare fibrewise topological spaces over B. If X is fibrewise regular, then so is Y.

proof:

Let $y \in Y_b$, where $b \in B$ and let V be an open set containing y in Y_b pick $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is an open set containing x , Since X is fibrewise regular , there exists an open set W of b and an open set U^* containing x in X_b such that $X_b \cap cl(U^*) \subset U$ .Then $Y_b \cap \varphi(cl(U^*) \subset \varphi(U) = V$. Since φ is closed, then $\varphi(cl(U^*)) = cl(\varphi(U^*))$ and since φ is open , then $\varphi(U^*)$ is an open set containing y. Thus Y_b is a fibrewise regular. ■

Definition: 2-2-22

A fibrewise topological space (X,τ) over B is called fibrewise T_3-space iff (X , τ) is fibrewise regular ,T_1-space .



Fibrewise T_3-space

Example: 2-2-23

Let X = [-2π , 2 π] with the cofinite topology and B = [-1 , 1] with the trivial topology , let p (x) = sin x be the projection map for any b ∈ [-1 ,1] , X_b = { x ∈ X : sin x = b } is finite subset of X . So X_b is a discrete space for all b ∈ B , and therefore X_b is fibrewise T_3-space , that is X is a fibrewise T_3-space over B . However (X,τ) is not T_3-space.

Theorem: 2-2-24

If (X, τ) is a T_3-space and is fibrewise space over B .Then (X ,τ) is a fibrewise T_3-space over B .

Proof:

Since every subspace of a T_3-space is a T_3-space, so it follows that every fibre subspace is T_3, and hence (X,τ) is fibrewise T_3- space.■

Theorem: 2-2-25

Let (X,τ)be a fibrewise T_3-space over B, then (X, τ) is fibrewise T_2- space .

Proof:

Let x_1, x_2 be distinct points of X_b, where $b \in B$, since X is fibrewise T_3-space, then there is open set U containing x_1 or x_2 in X_b , let U containing x_1 but not x_2 , since X is T_1-space , so $F = X_b \backslash U$ is closed set containing x_2 but not x_1 in X_b . Using definition of T_3-space, we get two disjoint open sets G, H such that $x_1 \in G_1$, $x_2 \in H$ in X_b, thus (X, τ) is T_2- space **.** ■

Definition: 2-2-26

A fibrewise topological space (X , τ) is said to be fibrewise completely regular if every non-trivial fibre subspace is completely regular , that is for each point $x \in X_b$, where $b \in B$, and for each closed set A in X_b $x \notin A$,there exists a continuous function $f_b : X_b \to I$ such that $f_b(x) = 0$, $f_b(A) =1$.

Remark: 2-2-27

1) If X is fibrewise completely regular space over B, then X^*_B is fibrewise completely regular space over B^*, for each subspace B^* of B.

2) Subspaces of fibrewise completely regular space are fibrewise completely regular spaces.

Theorem: 2-2-28

If (X, τ) is a completely regular and (X,τ) is fibrewise space over B, then (X, τ) is fibrewise completely regular space over B .

Proof:

Since every subspace of completely regular space is completely regular, so it follows that every fibre subspace is completely regular , and thus (X ,τ) is fibrewise completely regular space over B . ■

Proposition: 2-2-29

Let $\varphi : X \rightarrow X^*$ be a fibrewise embedding,where X and $X^*$are fibrewise topological spaces over B . If X^* is fibrewise completely regular , then so is X .

Proof:

The proof is similar to the proof of proposition (2-2-19), so it is omitted . ■

The class of fibrewise completely regular spaces is fibrewise multiplicative in the following sense .

Theorem: 2-2-30

If (X_α,τ_α) is fibrewise completely regular space over B, with projection P_α for all $\alpha \in \Lambda$. Then $X=\prod_{\alpha\in\Lambda} X_\alpha$ is a fibrewise regular topological space over B with projection $P_\beta \circ \pi_\beta$ for any $\beta \in \Lambda$.

Proof:

Suppose X_α is a fibrewise completely regular space overB,for all $\alpha \in \Lambda$ and with projection $P_\alpha : X_\alpha \to B$,then for any $\beta \in \Lambda$, $P_\beta \circ \pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to B$ is continuous .So $\prod_{\alpha\in\Lambda} X_\alpha$ is a fibrewise topological space over B , with projection $P_\beta \circ \pi_\beta$ such that $X_b = (P_\beta \circ \pi_\beta)^{-1}(b)$ for any $b \in B$, then $X_b = (P_\beta \circ \pi_\beta)^{-1}(b) = \pi_\beta^{-1}(P_\beta^{-1}(b)) = \{ (x_\alpha)_{\alpha\in\Lambda} : x_\alpha \in X_\alpha ,\ x_\beta \in p^{-1}(b) \}$ is a subspace of $\prod_{\alpha\in\Lambda} X_\alpha$ and since the product of completely regular spaces is a completely regular space , and every subspace of a completely regular space is a completely regular space , so it follows that every fibre subspace is a completely regular space and hence $\prod_{\alpha\in\Lambda} X_\alpha$ is fibrewise completely regular space . ■

Proposition: 2-2-31

Let $\varphi : X \to Y$ be an open ,closed and fibrewise surjection ,where X and Y are fibrewise topological spaces over B. If X is fibrewise completely regular, then so is Y.

Proof:

Let $y \in Y_b$,where $b \in B$ and let V_y be an open set containing y, pick $x \in X_b$, so that $V_\alpha = \varphi^{-1}(V_y)$ is an open set containing x .Since X is fibrewise completely regular there exists a nbhd W of b and an open set U_x containing x in X_W and a continuous function $\Omega : X_W \to I$ sach that $Y_b \cap U_y \subset \Omega^{-1}(0)$ and $X_w \cap (X_w - V_x) \subset \Omega^{-1}(1)$, and $Y_w \cap (Y_w - V_y) \subset \Omega^{-1}(1)$. ■

Definition: 2-2-32

A fibrewise topological space (X,τ) over B is called fibrewise $T_{3\frac{1}{2}}$-space iff (X,τ) is fibrewise completely regular, T_1-space.

Definition: 2-2-33

A fibrewise topological space X over B is called fibrewise normal if each non-trivial subspace of X is normal , that is for $b \in B$ and disjoint closed sets H and K of X_b , there exist a pair of disjoint open sets U,V containing H and K respectively.

Example: 2-2-34

If $X= \mathbb{R} = B$ with τ_1, τ_2 are the usual topology, $P:(X, \tau_1)\to(B , \tau_2)$ is defined by $P(x) = 1$ for all $x \in \mathbb{R}$, then

$$X_b = P^{-1}(b) = \begin{cases} X & \text{if } b = 1 \\ \emptyset & \text{if } b \neq 1 \end{cases}$$

Then X is fibrewise normal space over B .

Remark: 2-2-35

If X is fibrewise normal space over B , then X_{B*} is fibrewise normal space over B^* for each subspace B^* of B .

Proposition: 2-2-36

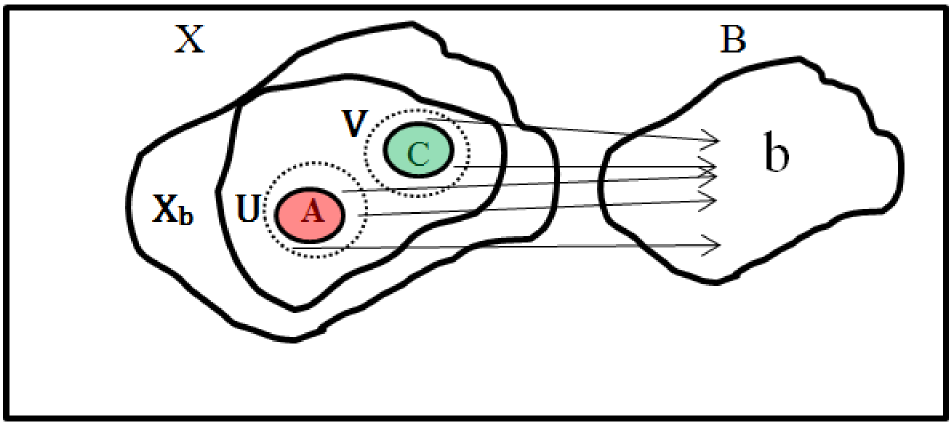
Let $\varphi : X \to X^*$ be a closed fibrewise embedding , where X and X^* are fibrewise topological spaces over B. If X^* is fibrewise normal, then so is X .

Proof:

Let b be a point of B and let H , K be disjoint closed sets of X_b , then φ (H) ,φ (K) are disjoint closed sets of X^*_b , since X^*_b is fibrewise normal there exists disjoint open sets U,V of X^*_b containing φ (H) , φ(K) in X^*_b . Then φ^{-1}(U) and φ^{-1}(V) are disjoint open sets of X_b containing H and K . ■

Definition: 2-2-37

A fibrewise topological space over B is called fibrewise T_4-space iff (X, τ) is fibrewise normal, T_1-space.



Fibrewise - T_4- space

Theorem: 2-2-38

a) Closed subspaces of fibrewise T_{4}-spaces are fibrewise T_{4}-spaces.

b) Every fibrewise T_4-space is fibrewise T_3-space.

c) A product of fibrewise normal spaces is not necessarily fibrewise normal.

Theorem: 2-2-39

If (X, τ) is a fibrewise topological space over B and p : X → B the Projection map is bijective, then every fibre subspace is trivial i-e X_b, is empty or a one-point set .

Corollary: 2-2-40

If (X, τ) is a fibrewise topological space over B and p: X → B is bijective , then (X , τ) is a T_i-fibrewise topological space for all i = 0,1,2,3,4

Definition: 2-2-41

(X, τ) is fibrewise completely normal over B iff every non-trivial fibre subspace is completely normal ; that is for every pair A , B of separated subsets of X_b , there are disjoint open sets U and V of X_b , with A $\subset$U and B $\subset$V.

Example: 2-2-42

Let X = B = {a, b, c} with τ_X = { ϕ, X, {a}, {b,c}} and τ_B is the trivial topology , p : X $\rightarrow$ B is the identity projection , then (X ,τ) is fibrewise completely normal over B .

Theorem: 2-2-43

If (X,τ) is completely normal space and (X,τ) is a fibrewise topological space over B, then (X,τ) is a fibrewise completely normal topological space over B.

Theorem / 2-2-44

A fibrewise topological space (X ,τ) over B is fibrewise completely normal topological space over B iff every subspace is fibrewise normal space over B .

Proof:

⟹: let X be a fibrewise completely normal space over B , and let Y be a subspace of X .Let A , C ⊆Y be disjoint closed subsets. Then clearly $\overline{A} \cap C = A \cap C = A \cap \overline{C} = \emptyset$, so by complete normality there are disjoint open sets U , V ⊆ X , such that A ⊆ U , C ⊆ V. Taking U ∩ Y and V∩ Y,we have disjoint open sets in the subspace topology Y containing A and C, respectively inY. It follows that Yis fibrewise normal space over B.

⟸: Suppose every subspace of X is fibrewise normal space over B , and let A , C ⊆ X be separated subsets , so that $\overline{A} \cap C = A \cap \overline{C} = \emptyset$. Let Y be subspace containing $\overline{A}$ and $\overline{C}$. Since Y is normal , there are disjoint open sets U ,V ⊆ Y such that $\overline{A} \subseteq U$ and $\overline{C} \subseteq V$. Then since $A \subseteq \overline{A}$ and $C \subseteq \overline{C}$, it follows that X is fibrewise completely normal space over B .■

Theorem: 2-2-45

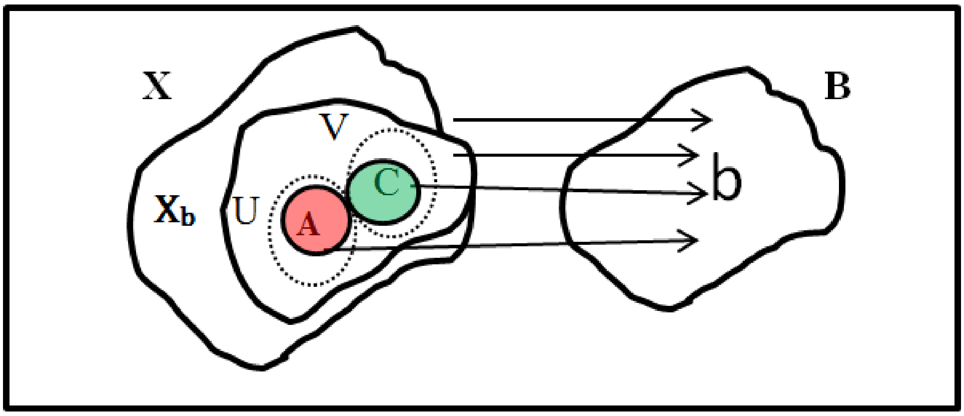
A subspace of fibrewise completely normal space over B is fibrewise completely normal space over B .

Proof:

By definition every subspace of X is normal, the same holds for every subspace A of X (as a subspace of A is also one of X, hence normal).

Definition: 2-2-46

A fibrewise topological space (X, τ) over B is called T_5-space iff (X , τ) is fibrewise completely normal T_1-space .



Fibrewise - T_5- space over B

Example: 2-2-47

Let X = B = ℝ with τ_1, τ_2 are the usual topology of the real line , P: (X, τ_1)→ (B , τ_2) is the identity projection function , then (X, τ_1) is fibrewise topological space over B . Consider the two open intervals $A = (0, \frac{1}{2})$ and $C = (\frac{1}{2}, 1)$. The sets do not intersect : $A \cap C = \emptyset$, but the closures , $\bar{A} = [0, \frac{1}{2}]$, $\bar{C} = [\frac{1}{2}, 1]$ with $\bar{A} \cap \bar{C} = \{\frac{1}{2}\}$. Nevertheless A and C are separated because $\bar{A} \cap C = \emptyset = A \cap \bar{C}$. A and C have the T_5 property because A and C themselves are disjoint open sets.

Theorem: 2-2-48

Every subspace of fibrewise T_5- space over B is fibrewise T_5- space over B .

Proof:

Since every fibrewise T_5-space over B is fibrewise completely normal T_1-space over B , by theorem (2-2-46) , and every subspace of fibrewise T_1-space over B is fibrewise T_1-space over B . So every subspace of fibrewise T_5- space over B is fibrewise T_5-space over B .■

By the above theorems we write the following result :-

Result: 2-2-49

i. Every subspace of fibrewise T_5- space over B is fibrewise T_4- space over B .
ii. Every fibrewise T_5- space over B is fibrewise T_4- space over B .

Chapter three

Fibrewise ideal topological spaces

If I is an ideal on a topological space (X,τ), a topology on X can be constructed called an ideal topology induced by the ideal I and, denoted by the * -topology or τ^*(I) or τ^*(I , τ).

The triple (X,τ^*,I) or the pair (X,τ^*(I)) are called ideal topological space, and if (X ,τ) is a fibrewise topological space over B ,then (X,τ^*(I)) or (X ,τ^*,I) is a fibrewise ideal topological space over B.

This chapter introduces the definition of fibrewise ideal topological space, studies some of their properties, and discusses the definition of fibrewise local function for a fibrewise topology with a fibrewise ideal.

I used the references **[1],[2],[3], [5], [6], [15] and [22]**

3-1 Fibrewise ideals

Definition: 3-1-1[2]

Let B be any set, and X be a fibrewise set over B, a non-empty collection I of subsets of X, is said to be fibrewise ideal on X, if it satisfies the following conditions:

i) If $A_1 \in$ I, and $A_2 \subset A_1$, then $A_2 \in$ I

ii) If $A_1 \in$ I, and $A_2 \in$ I, then $A_1 \cup A_2 \in$ I.

Examples: 3-1-2

i) Let $X = \{a, b, c\}$ be a fibrewise set over B, $I = \{\emptyset, \{a\}\}$, then I is fibrewise ideal on X.

ii) Let X be a non-empty fibrewise set over B, then $I = \{\emptyset\}$ is a fibrewise ideal on X over B.

iii) Let X be a non-empty fibrewise set over B, then $I = P(X)$ is a fibrewise ideal on X over B.

iv) The class $\{A \subseteq X : \forall x \in X_b, x \notin A\}$ where $b \in B$ is an ideal on the fibrewise set X which we denoted by I_b.

Lemma: 3-1-3 [2]

Let $\varphi: X \to Y$ be a fibrewise function, where X and Y are fibrewise topological spaces over B. Let I_b and J_b be two fibrewise ideals on X and Y respectively (for $b \in B$) then:

i) If φ is fibrewise surjection, then $J_b \subseteq \varphi(I_b)$.

ii) If φ is fibrewise bijection, then $\varphi(I_b) = J_b$.

Proof:

i) Let $E \subseteq Y$, and $E \in J_b$, then for every $y \in Y_b$, $y \notin E$. Since φ is a fibrewise surjection, then $\varphi(X_b) \subseteq Y_b$, implies for every $y \in \varphi(X_b)$, $y \notin E$. Thus for every $x \in X_b$, $x \notin \varphi^{-1}(E)$, implies $\varphi^{-1}(E) \in I_b$, then $\varphi^{-1}(J_b) \subseteq I_b$. Thus $J_b \subseteq \varphi(I_b)$.

ii) Let $\varphi : X \to Y$ be a fibrewise bijection and let $A \in \varphi(I_b)$, then $\varphi^{-1}(A) \in I_b$ implies for every $x \in X_b$,$x \notin \varphi^{-1}(A)$, then $\varphi(x) \notin A$, for all $x \in X_b$,since φ is onto,then for every $y \in Y_b$ there is $x \in X_b$, such that $\varphi(x) = y \notin A$ for every $y \in Y_b$, implies $A \in J_b$. So $\varphi(I_b) \subseteq J_b$ and from (i) thus $\varphi(I_b) = J_b$. ■

Lemma :3-1-4 [2]

Let Ψ:X→Y be a fibrewise injective, where X and Y are fibrewise topological spaces over B. If I is any fibrewise ideal on X, then $\Psi(I) = \{ \Psi(A) : A \in I\}$ is a fibrewise ideal on Y.

3-2 Fibrewise local function with respect to fibrewise ideal topology.

In this section we will define fibrewise ideal topology using fibrewise local function. First we give the following definition.

Definition: 3-2-1[2]

Let (X,τ) be a fibrewise topological space over B, with I as an ideal on X. Then for all $A \in P(X)$.$A^*(I,\tau)=\{x \in X: A \cap U \notin I$ for each neighborhood U of x $\}$ is called a fibrewise local function of A with respect to I and τ. We will write $A^*(I)$ or simply A^* for $A^*(I,\tau)$.

Examples: 3-2-2 [2]

i) If (X,τ) is any fibrewise topological space over B, $I = \{\emptyset\}$, then $A^* = cl(A)$ for any $A \subseteq X$.

ii) If (X,τ) is any fibrewise topological space over B, $I = P(X)$**,** then I is an ideal on X and $A^* = \emptyset$ for any $A \subseteq X$.

Using the results of [8],[13]

Lemma: 3-2-3

Let (X,τ) be a fibrewise topological space with I and J are ideals on X , and A and B be subsets of X . Then:

a) If $A \subseteq B$, then $A^* \subseteq B^*$

b) If $I \subseteq J$, then $A^*(J) \subseteq A^*(I)$

c) $A^* = cl(A^*) \subseteq cl(A)$ (A^* is closed subset of cl(A))

d) $(A \cup B)^* = A^* \cup B^*$

e) $A^* - B^* = (A-B)^* - B^*$

f) If $U \in \tau$, then $U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$

g) If $C \in I$, then $(A \cap C)^* = A^* = (A - C)^*$

h) $\emptyset^* = \emptyset$

i) $(A^*)^* = A^*$

Proof:

a) Suppose $A\subseteq B$, $x \notin B$, then there exists $U\in \tau$, such that $B \cap U\in I$. Since $A\cap U \subseteq B\cap U$, then $A\cap U\in I$. Hence $x \notin A^*$. Thus $A^*\subseteq B^*$.

b) Suppose that $x\notin A^*(I)$. Then there exists $U\in \tau$, such that $A\cap U\in I$. Since $I \subseteq J$, then $A\cap U\in J$ and $x\notin A^*(J)$. Therefore $A^*(J) \subseteq A^*(I)$.

c) We have $A^*\subseteq cl(A^*)$ in general . Let $x \in cl(A^*)$, then $A^*\cap U \neq \emptyset$ for every $U\in \tau$, $x \in U$.Therefore , there exists some $y\in A^*\cap U$ and $U\in \tau$, $y \in U$. Since $y\in A^*$, $A\cap cl(U)\notin I$ and hence $x\in A^*$. Hence $x \in A^*$. Hence we have $cl(A^*)\subseteq A^*$ and hence $A^*= cl(A^*)$, Again . Let $x \in A^*= cl(A^*) = \{ x \in X : U \cap A \notin I$ for any open set contains $x\}$ $\subset\{ x \in X : U \cap A \neq \emptyset \} = cl(A)$.

d) It follows from **(a)**, **(b)** and **(c)** that $A^*\cup B^*\subset (A\cup B)^*$.To prove the reverse inclusion , let $x \notin A^*\cup B^*$. Then x belongs neither to A^*, nor to B^*. Therefore there exists $U_x ,V_x \in \tau$ such that $U_x\cap A\in I$ and $V_x\cap B \in I$ since I is additive , then $(U_x\cap A) \cup (V_x\cap B) \in I$. Moreovere since I is hereditely and $(U_x \cap A) \cup (V_x \cap B) = (U_x\cap V_x) \cap (A\cup B)=(U_x \cap V_x \cap A) \cup (U_x\cap V_x \cap B) \subset (U_x\cap A) \cup (V_x\cap B) \in I$. So $(U_x \cap V_x) \cap (A\cup B) \in I$, since $(U_x\cap V_x) \in \tau$, so $x \in (A\cup B)^*$. Hence we abtain $A^*\cup B^* = (A\cup B)^*$.

e) We have by **(d)** $A^*= [(A\text{-}B) \cup (A\cap B)]^* = (A\text{-}B)^* \cup (A\cap B)^*\subseteq (A\text{-}B)^*\cup B^*$. Thus $A^*- B^* \subseteq (A\text{-}B)^*- B^*$. By **(a)**, **(b)** and **(c)**,

$(A\text{-}B)^* \subseteq A^*$ and hence $(A\text{-}B)^* - B^* \subseteq A^* - B^*$. Hence $A^* - B^* = (A - B)^* - B^*$.

f) If $U \in \tau$, then $U \cap A^* = U \cap \{x \in X : A \cap V \notin I$ for any nbhd V of $x\}$ $= \{x \in U : A \cap V \notin I$ for any nbhd V of $x\} = \{x : U \cap A \cap V \notin I$ for any nbhd V of $x\} = U \cap (U \cap A^*) \subseteq U \cap A^*$.

g) Since $C \in I$, by **(a)** , **(b)** and **(c)** , $C^* = \emptyset$. By **(e)** $A^* = (A\text{-}C)^*$ and by **(d)** $(A \cup C)^* = A^* \cup C^* = A^*$

h) Since $\emptyset^* = \{x \in X : \emptyset \cap U \notin I\} = \emptyset$, then $\emptyset^* = \emptyset$

i) $(A^*)^* = \{x \in X : U \cap A^* \notin I, U \in \tau\}$ and since $U \cap A^* \subseteq A^* \subseteq \{x \in X : A \cap U \notin I\} = A^*$, then $(A^*)^* \subseteq A^*$.■

Now we are ready to define fibrewise ideal topology by using the following proposition.

Proposition :3-2-4

Let (X,τ) be a fibrewise topological space over B, and I is an ideal on X. Then we define a map $cl^*(.) : P(X) \to P(X)$ by $cl^*A = cl^*(A)\ (I, \tau) = A \cup A^*(I, \tau)$ for all $A \in P(X)$.

The map $cl^*(\,)$ is a kuratowiski closure operator.

Corresponding to the ideal I on the fibrewise topological space (X,τ) , and so there exists a topology on X given by $\tau^*(I,\tau) = \{U \subseteq X : (cl^*U)^c = X \setminus U\}$. Which is finer than τ and called the fibrewise ideal topology induced on X, by the ideal I .

Proof:

Using theorem (3-2-3) it follows that :

i. If $A \subseteq B$, since $A^* \subseteq B^*$, $A \cup A^* \subseteq B \cup B^*$. So $cl^*(A) \subseteq cl^*(B)$.

ii. $cl^*(A \cup B) = (A \cup B) \cup (A \cup B)^*$, since $(A \cup B)^* = A^* \cup B^*$, so $(A \cup B) \cup (A \cup B)^* = (A \cup B) \cup A^* \cup B^* = A \cup A^* \cup B \cup B^* = cl^*(A) \cup cl^*(B)$.

iii. $cl^*(cl^*(A)) = cl^*(A \cup A^*) = (A \cup A^*) \cup (A \cup A^*)^* = A \cup A^* \cup A^* \cup (A^*)^*$, since $(A^*)^* = A^*$, so $A \cup A^* \cup A^* \cup A^* = A \cup A^* = cl^*(A)$.

iv. $cl^*(\emptyset) = \emptyset \cup \emptyset^* = \emptyset \cup \emptyset = \emptyset$.

Therefore the map is the kuratowiski closure operator and hence by theorem (1-2-4) it follows , there is a topology $\tau^*(I)$ or $\tau^*(I,\tau)$ induced by the ideal I and is finer than the topology τ , so $(X,\tau^*(I))$ or (X,τ^*,I) is a fibrewise ideal topological space.■

Definition: 3-2-5

If X is a fibrewise topological space and I is an ideal on X, then the topology defined by the above proposition is called the fibrewise ideal topology.

Remark: 3-2-6

Note that $\tau^*(I) = \tau^*(I,\tau) = \{U \subseteq X : (cl^*U)^c = X\backslash U\}$. Also $cl^*(A) = A \cup A^*(I,\tau)$, for any $A \subset X$.

Definition: 3-2-7

Let (X,τ^*, I) be a fibrewise ideal topological space over B . A subset A of X is said to be:

i. *-open or I-open if $A \in \tau^*(I)$.

ii. *-closed, or I-closed if it's complement is *-open (I-open) .

Examples: 3-2-8

1) Let X = B = {a, b, c, d}. Let $\tau_X = \tau_B = \{\emptyset, X, \{a\}, \{a, b\},\{a, b, d\}\}$. Define the identity projection $p : (X , \tau_X) \rightarrow (B , \tau_B)$; p(x) = x for each $x \in X$ .Then X is fibrewise topology and let I = {∅ ,{d}} is fibrewise ideal on X . Now let A = { c , d} , then A^c = {a , b} and $(A^c)^*$= {c}, then $cl^*(A^c) = A^c \cup A^* = \{c, d\}$, so $(cl^*(A))^c = A^c.\{a, b\}$.Thus A^c is *-open set in $\tau^*(I)$, implies A is *-closed .

2) let (X,τ) be a fibrewise topological space over B , with fibrewise ideal I on X , then :

i) If $I = \{\emptyset\}$, then $cl^*(A) = cl(A)$. So $\tau^*(I, \tau) = \tau$.

ii) If $I = \mathcal{P}(X)$, then $cl^*(A) = A$. So $\tau^*(I,\tau)$ = is the fibrewise discrete topology.

iii) If $X =\{a , b, c\}$, τ_X is fibrewise topology over $B =\{\emptyset , X, \{a\}, \{b,c\}\}$, I , J are fibrewise ideals on X over B. such that $I =\{ \emptyset ,\{a\}\}$, $J=\{\emptyset ,\{b\}\}$, and let $A = \{a , b\}$, then in $\tau^*(I)$, $A^*(I ,\tau) = \{b,c\}$, so $cl^*(A) = A\cup A^*=\{a ,b, c\}$, then $cl^*(A)\neq A$, then A is not I-closed (*-closed) . But in $\tau^*(J)$, $A^*(J,\tau)= \{a\}$, then $cl^*(A) = A\cup A^*= \{a , b\}$.Thus $cl^*(A) = A$, so A is J-closed (*-closed) set ; implies $A^c = \{c\}$ is J-open(*-open) set .Then $\{c\}$ is J-open (*-open) set in $\tau^*(J)$. But not I-open (*-open) set in $\tau^*(I)$.

using [14],[15] to write the following definition

Definition: 3-2-9

Let $(X, \tau^*(I))$ be fibrewise ideal topological space , and let $A\subset X$. Then a point $x\in X$ is called an I-limit for A iff each I-open set U containing x,

$U \cap (A\backslash\{x\}) \neq \emptyset$. The set of all I- limit points of A is denoted by I- limit A .

Example: 3-2-10

Let X = { a , b , c , d} , τ = { ∅, X ,{d} , {a , c}, { a , c , d}} and I = { ∅,{b}}, J = {∅, {c}} are fibrewise ideals on X , and let A={a , c} , then the I-open sets are{d}, {a, c}, {a, c, d} ,∅ , X then a is I- limit A . Because for each I-open set U contains a , $U \cap (A\backslash\{a\}) \neq \emptyset$, since $\{a , c\} \cap (\{a , c\}\backslash\{a\}) \neq \emptyset$, $\{a , c , d\} \cap(\{a , c\}\backslash\{a\}) \neq \emptyset$.

But a is not J-limit A. Because J-open sets are {a},{d},{a, c},{a,d}, {a, c, d}, ∅, X. Then there is a J-open set U such that $U \cap(A\backslash\{a\}) = \emptyset$ it is $\{a\}\cap (\{a, c\}\backslash\{a\}) =\emptyset$. Thus a is not J- limit A .

Definition :3-2-11

A map $f : X \to Y$ is called I-open (*-open) (resp I-closed (*-closed) if the image of each I-open (*-open) (resp I-closed (*-closed)) set in X is I-open (*-open) (resp I-closed (*-closed)) set in Y.

Lemma: 3-2-12 [2]

Let $\psi : (X ,\tau) \to (Y,\delta)$ be continuous fibrewise function , where X and Y are fibrewise spaces over B . Then $\psi : (X , \tau^*(I)) \to (Y, \delta)$ is a continuous fibrewise function for any ideal I on X .

Proof:

Since $\psi : (X, \tau) \rightarrow (Y, \delta)$ is continuous fibrewise function , then for every open set V in Y , $\Psi^{-1}(V)$ is an open in X , that is $\Psi^{-1}(V) \in \tau$.

Now consider $\psi : (X, \tau^*(I)) \rightarrow (Y, \delta)$ is fibrewise function , since $\tau^*(I)$ is finer than τ , then every open set in τ is in $\tau^*(I)$, so for every open set V in Y, $\Psi^{-1}(V)$ is an open set in X .Thus $\Psi(X, \tau^*(I)) \rightarrow (Y, \delta)$ is continuous fibrewise function .

Definition: 3-2-13

A map $f : X \rightarrow Y$ is called I-continuous (*-continuous) if the inverse image of each I-open(*-open) set in Y is I-open (*-open) in X .

Example: 3-2-14

A constant map $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is always continuous as map $f : (X, \tau_1^*(I)) \rightarrow (Y, \tau_2(J))$. Where I , J are any arbitrary two ideals on X and Y respectively.

Proposition: 3-2-15 [2]

Let $\psi : (X, \tau_1, I) \rightarrow (Y, \tau_2, \psi(I))$ be a continuous fibrewise injection, where X and Y are fibrewise spaces over B, and I be a fibrewise ideal on X. Then $\psi: (X, \tau_1^*, I) \rightarrow (Y, \tau_2^*, \psi(I))$ is I-continuous (*- continuous) fibrewise function .

Proof:

Let $U^c \in \tau_2^*(\psi(I))$ .Then $cl^*(U) = U = U \cup U^*$, so $U^*(\psi(I)) \subseteq U$ and $U^c \subseteq (U^*(\psi(I)))^c$. For any $y \in U^c$, there exists a neighborhood V of y such that $U \cap V \in \psi(I)$, so $\psi^{-1}(U \cap V) \in I$, and so $\psi^{-1}(U) \cap \psi^{-1}(V) \in I$.

Therefor $\psi^{-1}(y) \notin (\psi^{-1}(U))^*$ and $\psi^{-1}(y) \in (\psi^{-1}(U^*))^c$. Thus $\psi^{-1}(U^c) \subseteq (\psi^{-1}(U^*))^c$ and so $\psi(U)^* \subseteq (\psi^{-1}(U^c))^c = \psi^{-1}(U)$ and $cl^*(\psi^{-1}(U)) = \psi^{-1}(U) \cup (\psi^{-1}(U))^* = \psi^{-1}(U)$. Thus $(\psi^{-1}(U))^c \in \tau_1^*(I)$ but $\psi^{-1}(U^c) = (\psi^{-1}(U))^c$. Then $\psi^{-1}(U^c) \in \tau_1^*(I)$. Hence ψ is I-continuous (*-continuous) .■

Lemma: 3-2-16 [2]

Let $\varphi : (X, \tau, I) \to (Y, \delta, \varphi(I))$ be open continuous fibrewise function over B, where X,Y are fibrewise topological spaces over B,with a fibrewise ideal I on X . Then $\varphi^{-1}(E^*) = (\varphi^{-1}(E))^*$, for each subset $E \subseteq Y$.

Proof:

Let $x \in \varphi^{-1}(E^*)$. Then $\varphi(x) \in E^*$, so for every neighborhood V of φ (x) in Y , $V \cap E \notin \varphi(I)$. Since φ is open function , then for each neighborhood U of x in X , φ(U) is a neighborhood of φ (x) in Y, and also $\varphi(U) \cap E \notin \varphi(I)$ then $\varphi^{-1}(\varphi(u) \cap E) \notin I$, implies $U \cap \varphi^{-1}(E) \notin I$, for each neighborhood U of x in X . Hence $x \in (\varphi^{-1}(E))^*$. And therefore $\varphi^{-1}(E^*) \subseteq (\varphi^{-1}(E))^*$.

Conversely, let $x \in (\varphi^{-1}(E))^*$. Then for every neighborhood U of x in X, $U \cap \varphi^{-1}(E) \notin I$. Since φ is continuous function , then for every neighborhood V of $\varphi(x)$ in Y, $\varphi^{-1}(V) \cap \varphi^{-1}(E) \notin I$, then $\varphi(\varphi^{-1}(V) \cap \varphi^{-1}(E)) \notin \varphi(I)$, implies that $V \cap E \notin \varphi(I)$, for every neighborhood V of $\varphi(x)$ in Y, hence $\varphi(x) \in E^*$, then $x \in \varphi^{-1}(E^*)$. Thus $(\varphi^{-1}(E))^* \subseteq (\varphi^{-1}(E^*)$. ■

Proposition: 3-2-17 [2]

Let $\varphi : (X, \tau, I_b) \to (Y, \delta, J_b)$ be a continuous fibrewise bijection, where X and Y are fibrewise topological spaces over B, with fibrewise ideals I, J on X, Y respectively, then the following statements are equivalent :

1) $\varphi : (X, \tau^*(I_b)) \to (Y, \delta^*(J_b))$ is a homeomorphism.

2) $\varphi(A^*) = \varphi(A)^*, \forall A \subseteq X$.

Proof: (1) →(2)

Let $y \notin \varphi(A^*)$. Then $\varphi^{-1}(y) \notin A^*$, implies there is a neighborhood U of $\varphi^{-1}(y)$ in X, such that $A \cap U \in I_b$, then $\varphi(A \cap U) \in \varphi(I_b) = J_b$, and $\varphi(A) \cap \varphi(U) \in J_b$, since φ is fibrewise homeomorphism then $\varphi(U)$ is a neighborhood of y in Y and also $y \notin (\varphi(A))^*$ .So $(\varphi(A))^* \subseteq \varphi(A^*)$.

Conversely, let $y \notin (\varphi(A))^*$, then there is a neighborhood W of y in Y such that $\varphi(A) \cap W \in \varphi(J_b)$ then $A \cap \varphi^{-1}(w) \in I_b$, and $\varphi^{-1}(y) \in \varphi^{-1}(w)$ since φ is fibrewise homeomorphism then $\varphi^{-1}(w)$ is a neighborhood of $\varphi^{-1}(y)$ in X . Therefor $\varphi^{-1}(y) \notin A^*$. So $y \notin \varphi(A^*)$ .Hence $\varphi(A^*) \subseteq (\varphi(A))^*$. So finally we have $\varphi(A^*) = (\varphi(A))^*$.

(2) →(1) let $U^c \in \tau^*(I_b)$ then $U^* \subseteq U$ and $\varphi(U^*) \subseteq \varphi(U)$ by **(2)** , $(\varphi(U))^* = \varphi(U^*) \subseteq \varphi(U)$, and $\varphi(U) = \varphi(U) \cup (\varphi(U))^* = cl^*(\varphi(U))$, hence $(\varphi(U))^c \in \delta^*(J_b)$. Thus φ is open function . Similarly , we prove that $\varphi^{-1} : (Y, \delta^*(J_b)) \to (X, \tau^*(I_b))$ is an open function , and so φ is a homeomorphism. ■

3-3 Fibrewise local function over b∈B and the generated fibrewise topology over B on X_b:

Definition: 3-3-1 [2]

Let (X, τ) be a fibrewise topological space over B. with fibrewise ideal I on X , if $b \in B$ with $X_b \neq \emptyset$, then $A^*_b(I, \tau) = \{ x \in X_b : A \cap U \notin I,$ for each neighborhood U of x$\}$. Will be called fibrewise local function of A over b.

When there is no chance for confusion we will simply write $A^*_b(I)$ or A^*_b for $A^*_b(I,\tau)$. And we define the closure operator on X_b , to be $cl^*(A_b) = A_b \cup A^*_b$, for every $A \subseteq X$, where $A_b = A \cap X_b$, and hence it generates a new fibrewise topology on X_b over b to be $\tau^*_b = \{ A_b \subseteq X_b : (cl^* A_b)^* = (X_b \setminus A_b)\}$ which is finer than τ_{X_b}.

Notations: 3-3-2

If I is an ideal on X over B, then

i) $I_b = \{A \subseteq X : \forall x \in X_b , x \notin A\}$

ii) $I_{X_b} = \{ A \subset X_b : A \in I \} = \{A \cap X_b : A \in I \}$

iii) $I_{X_B} = \{ A \subseteq X_B : X = X_B = P^{-1}(B) \}$

Example: 3-3-3

Let (X, τ) be a fibrewise topological space over B. such that , $X = \{1,2,3,4\}$ $\tau = \{ \emptyset, \{4\},\{1,3\},\{1,3,4\}, X\}$ with a fibrewise ideal $I = \{ \emptyset, \{1\}\}$ on X , let $A_b = \{2\}$ for any $b \in B$, then $A_b^* = \{2\}$, implies $cl^*(A_b) = A_b \cup A^*_b = \{2\}$, so $cl^*(A_b) = A_b$, and so A_b is I-closed (*-closed) in $\tau^*(I)$, then $\tau_{X_b} \subseteq \tau^* (I_b)$. But if $I = \{ \emptyset\}$, then $A^*_b = cl(A_b)$, implies $cl^*(A_b) = cl(A_b)$, then $\tau^*(\{\emptyset\}_b) = \tau_{X_b}$.And if $I = p(X)$ on X, let $A_b = \{1,3\}$ for any $b \in B$, then $A_b^* = \emptyset$ and hence $cl^*(A_b) = A_b$, then $\tau^*(P(X))$ is the fibrewise discrete topology on X_b .

Lemma: 3-3-4

Let (X , τ) be a fibrewise topological space over B with fibrewise ideal I on X , then for every $A \subseteq X$ and any $b \in B$.

1) $A_b^* = X_b \cap A^*$

2) $\tau_b^*(I) = \{ X_b \cap G : G \in \tau^*(I) \} = \tau^*_{X_b}$.

Proof:

1) Since $A_b^* \subseteq X_b$ and $A_b^* \subseteq A^*$, then $A_b^* \subseteq X_b \cap A^*$.

Conversely, let $x \in X_b \cap A^*$, then $x \in X_b$ and $x \in A^*$, implies $x \in X_b$ and for each neighborhood U of x, $A \cap U \notin I$, so $x \in A_b^*$, then $X_b \cap A^* \subseteq A_b^*$. Hence $A_b^* = X_b \cap A^*$.

2) Let $(A_b^*)^c \in \tau_b^*(I)$, then $cl^*(A_b) = A_b$ and $cl^*(A_b) = A_b \cup A_b^*$, then

$A_b = A_b \cup A_b^*$, since $A_b = X_b \cap A$ and $A_b^* = X_b \cap A^*$ that's implies $A_b = (X_b \cap A) \cup (X_b \cap A^*) = X_b \cap (A \cup A^*) = X_b \cap cl^*(A)$, and so $A = \bigcup_{b \in B} A_b = \bigcup_{b \in B}(X_b \cap cl^*(A)) = X \cap cl^*(A) = cl^*(A)$,hence $A^c \in \tau^*(I)$. Then every element $(A_b^*)^c \in \tau_b^*(I)$ is an element in $\tau_{X_b}^*$.

Lemma: 2-3-5

Let (X,τ,I) be a fibrewise ideal topological space over B with fibrewise ideal I on X , then $X^*_b = X_b$, for all $b \in B$ iff $I \cap \tau = \{\emptyset\}$.

Proof:

$\Longrightarrow$: Let for each $b \in B$, $X_b^* = X_b$.Then for every $x \in X_b$, $U \cap X_b \notin I$, for each open set U containing x , then $U \notin I$ for every $U \in \tau$, that is implies $I \cap \tau = \{\emptyset\}$.

$\Longleftarrow$: Conversely, let $I \cap \tau = \{\emptyset\}$, then $X = X^*$ and so $X_b = X_b \cap X = X_b \cap X^* = X_b^*$. ■

Theorem: 3-3-6

Let (X,τ, I) be a fibrewise ideal topological space over B with a fibrewise ideal I on X, and $A \subseteq X$ then :

i) $\cup\{A_b^*; b\in B\} = (\cup\{A_b; b\in B\})^* = A^*$

ii) $\cap\{A_b^*; b\in B\} = (\cap\{A_b; b\in B\})^* = \emptyset$

Proof:

i) Since $A_b^* = X_b \cap A^*$, then $\cup A_b^* = \cup(X_b \cap A^*) = \cup(X_b \cap \{x\in X: U \cap A \notin I$ for any nbhd U of x $\}) = \cup\{x \in X_b \cap X: U \cap A \notin I$ for any nbhd U of x $\} = \cup\{x \in X_b : U \cap A \notin I$ for any nbhd U of x $\} = \cup\{A_b : b\in B\}^* = (\cup\{A_b : b\in B\})^* = A^*$

ii) $\cap A_b^* = \cap(X_b \cap A^*) = \cap(X_b \cap \{x\in X : U \cap A \notin I$ for any nbhd U of x $\} = \cap\{x \in X_b \cap X, U \cap A \notin I$ for any nbhd U of x $\} = \cap\{x\in X_b \cap X : U \cap A \notin I$ for any nbhd U of x $\} = \cap\{A_b^* : b\in B\} = (\cap\{A_b : b\in B\})^* = \emptyset$.■

Chapter four

Separation axioms in fibrewise ideal topological spaces

The aim of this chapter is to study separation axioms in fibrewise ideal topological spaces. particularly, define T_0-spaces, T_1-spaces, T_2-spaces, T_3-spaces , T_4-spaces , and T_5-spaces in fibrewise spaces in the context of ideal topological spaces .

In addition, to discuss some of the operations of separation axioms, products of fibrewise ideal topological spaces , and some theorems of continuous fibrewise functions on separation axioms in fibrewise ideal topological spaces.

4-1 Preliminary

In this section I used definitions in [11] , [12] and [19]

Definition: 4-1-1

A fibrewise ideal I is said to be:

1) Fibrewise condense or τ- fibrewise boundary if $\tau \cap I = \{\emptyset\}$
2) Fibrewise condense if Po(X) $\cap$ I = $\{\emptyset\}$, where Po(X) is the family of all open sets in a fibrewise ideal topological space (X, τ^*(I))

Notation: 4-1-2

The set of all open sets of a fibrewise ideal space (X , τ^*(I)) over B containing a point x$\in$ X is denoted by Io(X ,x).

In the section we used [4], [5], [9], [10], [11],[12], [14], [19], and [21] to define sepanation axioms in fibrewise ideal

4-2 Fibrewise ideal T_0- topological spaces:

Definition: 4-2-1

A fibrewise ideal topological space $(X,\tau^*(I))$ over a topological space B is said to be fibrewise T_0-space if every non-trivial fibrewise ideal subspace is T_0-space i-e, for any distinct pair of points in X_b, there is an I-open (*-open) set containing one of the points but not the other.

Example: 4-2-2

If $X = \mathbb{R}$, τ_1, is the cofinite topology , and $B = \mathbb{R}$, τ_2 is the trivial topology , and $p : (X, \tau_1) \to (B, \tau_2)$, $p(x) = x^2$, and $I =\{ A\subseteq \mathbb{R}$, A is a finite subset of $\mathbb{R}$ } . Then for any $A\subseteq \mathbb{R}$,

$$A^*(I, \tau)=\begin{cases} \bar{A}, & \text{if } A \text{ is infinite} \\ \emptyset, & \text{if } A \text{ is finite} \end{cases}$$

So $cl^*(A) = A \cup A^*(I, \tau) = \bar{A}$, so $\tau^* = \tau_1$, the cofinite topology .

But for any $b\in B$, $X_b=\begin{cases} \{-\sqrt{b}, \sqrt{b}\} & \text{if } b > 0 \\ \{0\} & \text{if } b = 0 \\ \emptyset & \text{if } b < 0 \end{cases}$

Hence every non-trivial fibrewise ideal subspace is discrete and so $(\mathbb{R}, \tau^*(I))$ is a fibrewise ideal T_0-space over B .

Theorem: 4-2-3

Let (X, τ) be a fibrewise topological space over B, and I is fibrewise ideal space on X , then (X , τ^*(I)) is fibrewise T_0-space iff for each pair of distinct point x , y of X $\overline{\{x\}} \neq \overline{\{y\}}$

Proof:

⇒: Let (X , τ^*(I)) be a fibrewise T_0-space over B , and x , y be two distinct points in X . Then there exists an I-open set U containing x does not containing y , or there exists an I-open set containing y and does not containing x . Let x ∈ U, y∉ U, where U is I-open. Then X\U is a closed set contains y and does not contain x . Thus $\overline{\{y\}} \subset X\backslash U$, $x \notin \overline{\{y\}}$ .Thus $\overline{\{x\}} \neq \overline{\{y\}}$

⇐: Let x , y be distinct points in X_b ,where b∈ B and $\overline{\{x\}} \neq \overline{\{y\}}$. Then there exists at least one point of X belong to any one of the two sets, and not the other, let $x \in \overline{\{x\}}$, $x \notin \overline{\{y\}}$. So $x \in X\backslash\overline{\{y\}}$, since $X\backslash\overline{\{y\}}$ is an open set does not contain y . So (X,τ^*(I)) is a fibrewse T_0-space over B . ■

Theorem: 4-2-4

Every subspace of fibrewise ideal T_0-space over B is a fibrewise ideal T_0-space over B.

Proof:

Let Y_b be a non-trivial subspace of a fibrewise ideal T_0-space $(X, \tau^*(I))$ over B and x , y be two distinct points of Y_b . Then either there exists an I-open set U in X such that $x\in U$ and $y\notin U$ or there exists an I-open set V in X such that $x \notin V$ and $y \in V$, then either $U \cap Y_b = U_b$ is I-open in Y_b with $x \in U_b$, $y\notin U_b$ or $V \cap Y_b = V_b$ is an I-open in Y_b with $y \in V_b$, $x\notin V_b$. Hence $(Y, \tau^*_Y(I_Y))$ is a fibrewise ideal T_0-space over B . ■

Definition: 4-2-5

Let (X , τ) , (Y, δ) be two fibrewise topological spaces over B , and I be a fibrewise ideal on X , a function $f : (X , \tau^*(I)) \to (Y , \delta)$ is said to be point fibrewise I-closure one-to-one iff for $x , y \in X_b$, $b \in B$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, then $cl(\{f(x)\}) \neq cl(\{f(y)\})$.

Theorem :4-2-6

Let (X, τ) and (Y, δ) be two fibrewise topological spaces over B and I is fibrewise ideal on X .

If $f : (X , \tau^*(I))\to(Y,\delta)$ is a point fibrewise I-closure one-to-one and $(X, \tau^*(I))$ is fibrewise ideal T_0-space over B , then f is one-to-one .

Proof:

Since (X, τ^*(I)) is fibrewise ideal T_0-space over B , then $\overline{\{x\}} \neq \overline{\{y\}}$ for any $x \neq y$ in X_b , b∈B . But *f* is point I-closure one-to-one implies cl({*f* (x)})≠ cl({*f* (y)}) , and so *f* (x) ≠*f* (y) .Thus *f* is one-to-one . ■

Theorem: 4-2-7

If *f* : (X , τ^*(I)) → (Y,δ) is a function from fibrewise ideal T_0-space (X , τ^*(I)) over B into a fibrewise topological space (Y, δ) over B .Then *f* is point I-closure one-to-one iff *f* is one-to-one.

Proof:

⟹: By Theorem (4-2-6) it is clear if *f* I- closure one-to-one , then *f* is one-to-one.

⟸ : Assume *f* : (X , τ^*(I)) → (Y,δ) is one-to-one such that (X , τ^*(I)) is fibrewise ideal T_0-space over B , and (Y, δ) is fibrewise topological space over B , for each pair of distinct points x ,y of X_b, b∈B , then *f* (x) ≠*f* (y) since (X , τ^*(I)) is fibrewise ideal T_0-space over B, by Theorem (4-2-3) $\overline{\{x\}} \neq \overline{\{y\}}$,and so cl({*f*(x) }) ≠ cl ({*f*(y)}).This implies *f* is point I-closure one-to-one . ■

Theorem :4-2-8

Let (X, τ) and (Y, δ) be two fibrewise topological spaces over B and I is fibrewise ideal on X and J is fibrewise ideal on Y .

If f : (X, τ^*(I))→ (Y,δ^*(J)) is a fibrewise injective continuous , function , and Y is fibrewise ideal T_0-space over B . Then (X, τ^*(I)) is fibrewise ideal T_0-space over B.

Proof:

Let x and y be any two distinct points of X_b , b∈ B , since f is fibrewise injective and Y is fibrewise ideal T_0-space over B , there exists an I-open set U_x in Y such that f (x) ∈ U_x and f (y) ∉ U_x or there exists an I-open set U_y in Y such that f (y) ∈ U_y and f (x) ∉ U_y , with f (x) ≠ f (y) . By fibrewise I-continuonity of f , then $f^{-1}(U_x)$ is I-open set in (X , τ^*(I)) , such that x ∈ $f^{-1}(U_x)$ and y∉ $f^{-1}(U_x)$ or $f^{-1}(U_y)$ is I-open set in (X ,τ^*(I)), such that y ∈ $f^{-1}(U_y)$ and x∉ $f^{-1}(U_y)$ .Thus (X, τ^*(I))is a fibrewise idealT$_0$- topological space over B .■

4-3- Fibrewise ideal T_1- topological spaces:

Definition: 4-3-1

A fibrewise ideal topological space (X, τ^*(I)) over B is a fibrewise ideal T_1-space over B if every non-trivial fibre subspace is T_1-space i-e, for any distinct points x , y of X_b , there exists a pair of I-open sets in X_b one containing x but not y and the other containing y but not x .

Example :4-3-2

Let $X = \mathbb{R}$, τ_1 is the co-countable topology , $B = \mathbb{R}$, τ_2 is the trivial topology , and $I = \{ A \subseteq \mathbb{R} : A \text{ is countable}\}$. $p : (X , \tau_1) \to (B , \tau_2)$ defined by $p(x) = \begin{cases} 1 , & \text{if } x \in Q \\ 0 , & \text{if } x \notin Q \end{cases}$

Then p is continuous and for any $b \in B$

$$X_b = p^{-1}(b) = \begin{cases} Q , & \text{if } b = 1 \\ Q^c , & \text{if } b = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Then X_b is a discrete ideal subspace of $(\mathbb{R}, \tau_1)$ if b =1 and X_b is a co-countable ideal subspace of $(\mathbb{R}, \tau_1)$ if b = 0 .Hence every non-trivial fibre subspace is a T_1-space , so $(X , \tau^*(I))$ is a fibrewise ideal T_1-space over B.

Theorem: 4-3-3

If a fibrewise ideal topological space $(X, \tau^*(I))$ over B is fibrewise T_1-space over B, then each one point set is I-closed in X.

Proof:

Let $(X, \tau^*(I))$ be a fibrewise T_1-space over B, and let $x \in X_b$ if $y \in X_b$ with $y \neq x$, there exists two I-open sets such that $x \in U_x$, $y \notin U_x$, and

$x \notin V_y$, $y \in V_y$. Hence $y \in V_y \subset X \backslash \{x\}$. So $X \backslash \{x\}$ is a union of I-open sets. Then $\{x\}$ is I-closed ■

Theorem :4-3-4

Let X be a fibrewise T_1-space over B, and $f : (X, \tau) \to (Y, \delta^*(J))$ is an I-closed surjective function . Then $(Y, \delta^*(I))$ is fibrewise ideal T_1-space over B.

Proof:

Suppose $y \in Y$. Since f is surjective, there exists a point $x \in X$ such that $y = f(x)$. Since X is fibrewise T_1-space over B , $\{x\}$ is closed in X. Again by hypothesis, $f(\{x\}) = \{y\}$ is I-closed in Y. Hence Y is fibrewise ideal T_1-space over B. ■

Theorem :4-3-5

If $(X, \tau^*(I))$ is fibrewise infinite T_1-space over B, and $x \in$ I-limit A for some $A \subset X$, then every I-neighborhood of x contains infinitely many points of A .

Proof:

Let $x \in$ I-limit A and suppose U is a I-neighborhood of x, such that $U \cap A$ is finite . let $U \cap A = \{x_1, x_2, x_3 \ldots\ldots\ldots x_n\} = C$. Clearly C is closed set . Hence $V = (U \cap A) \backslash (C \backslash \{x\})$ is I-neighborhood to the point x and $V \cap (A \backslash \{x\}) = \emptyset$ which implies that $x \notin$ I-limit A , which is a contradiction to our assumption.Therefore the given statement in the theorem is true . ■

Example: 4-3-6

Let X = { a, b , c, d} ,τ ={ X, ∅ , {b}, {a, b} , { b,c} ,{ a,b,c},{a,b,d}} is fibrewise topological space over B , with fibrewise ideal I = { ∅, {d}} , and let A = { a,b} then a is I-limit A , since every I-open U containing a, U∩({a,b}\{a}) ≠ ∅. But b is not I-limit A, since there is I-open {b} containing b such that { b}∩ ({a,b}\{b}) = ∅

Theorem: 4-3-7

Let (X,τ^*(I)) and (Y, δ^*(J)) be two fibrewise ideal topological spaces over B, and *f* : (X , τ^*(I)) →(Y,δ^*(J)) be an injective and I-continuous function . If (Y, δ^*(J)) is fibrewise T_1-space over B , then (X , τ^*(I)) is fibrewise T_1-space over B.

Proof:

The proof is similar to the proof of Theorem **(**4-2-10). ■

Theorem :4-3-8

In the fibrewise ideal topological space (X ,τ^*(I)) over B , if X is fibrewise ideal T_1-space over B . Then X is fibrewise ideal T_0-space over B.

Proof:

Let (X, τ^*(I)) be a fibrewise ideal T_1-space over B , and let x , y ∈ X_b , where b ∈ B such that x≠ y , since X_b is fibrewise ideal T_1-space , then there exists open sets U ,V such that U containing x but not y and V containing y but not x , then X is fibrewise ideal T_0-space over B. ■

4-4 Fibrewise ideal T_2''Hausdorff" topological spaces:

Definition: 4-4-1

A fibrewise ideal topological space (X , τ^*(I)) over B is said to be fibrewise ideal T_2" Hausdorff "-space if every non- trivial fibrewise ideal subspace is T_2-space i-e , for each pair of distinct points x , y of X_b , there exists a pair of disjoint open sets in X_b , one containing x and the other containing y where b ∈ B .

Example: 4-4-2

Let X = ℝ, τ_1 the usual topology on ℝ, B = ℝ with τ_2 the trivial topology , if p : (ℝ, τ_1) → (ℝ, τ_2) defined by p(x) = x^2

let I = { A⊆ ℝ, A is a finite subset of ℝ } .

Then A^*(I , τ) = { x ∈ ℝ: U∩A ∉ I for any open set U containing x} =

$$\begin{cases} \bar{A} & \text{if } A \text{ is infinite set} \\ \emptyset & \text{if } A \text{ is a finite set} \end{cases}$$

So $\tau^* = \tau$ is the usual topology

$$\text{For } b \in B,\ X_b = p^{-1}(b) = \begin{cases} \{-\sqrt{b}, \sqrt{b}\} & \text{if} \quad b > 0 \\ \{0\} & \text{if} \quad b = 0 \\ \emptyset & \text{if} \quad b < 0 \end{cases}$$

So every non-trivial subspace is a T_2-space .

Thus (ℝ, τ^*(I)) is fibrewise ideal T_2-space over B .

Theorem: 4-4-3

If (X,τ) is a fibrewise T_i-space over B , then $(X ,\tau^*(I))$ is a fibrewise ideal T_i-space over B for $i = 0,1,2$.

Proof:

The proof is obvious, since every $\tau^*(I)$ is finer than τ and the result follows from Theorem (1-5-27). ■

Theorem: 4-4-4

Let (X, τ), (Y,δ) be two fibrewise topological spaces over B and I, J are fibrewise ideals on X ,Y respectively .

If $f : (X , \tau^*(I)) \to (Y, \delta^*(J))$ is injective open and continuous, and $(Y, \delta^*(J))$ is fibrewise ideal T_2-space , then $(X , \tau^*(I))$ is fibrewise ideal T_2-space over B.

Proof:

Since f is injective, $f(x) \neq f(y)$ for each $x , y \in X_b$, and $x \neq y$. Now $(Y, \delta^*(J))$ being fibrewise ideal T_2-space , there exists I-open sets G , H in Y_b , where $b \in B$, such that $f(x) \in G$, $f(y) \in H$, and $G \cap H = \emptyset$. Let $U = f^{-1}(G)$ and $V = f^{-1}(H)$ .Then by hypothesis , U and V are I-open sets in X_b . Also, $x \in f^{-1}(G) = U$, $y \in f^{-1}(H) = V$, and $U \cap V = f^{-1}(G) \cap f^{-1}(H) = \emptyset$ .Hence $(X ,^*(I))$ is fibrewise ideal T_2-space over B .■

Corollary: 4-4-5

Let $(X , \tau^*(I))$, $(Y,\delta^*(J))$ be two fibrewise ideal topological spaces over B, If $f:(X,\tau^*(I)) \to (Y,\delta^*(J))$ is injective and I-closed and $(Y,\delta^*(J))$ is fibrewise ideal T_2-space over B , then $(X , \tau^*(I))$ is fibrewise ideal T_1-space over B .

Theorem :4-4-6

If $f : (X , \tau^*(I)) \to (Y,\delta^*(J))$ is I-continuous , $(Y, \delta^*(J))$ is fibrewise ideal T_2-space over B, then the set $\{ (x_1,x_2) : f(x_1) = f(x_2) \}$ is I- closed in $X\times X$.

Proof:

Let $A = \{ (x_1, x_2) : f(x_1) = f(x_2) \}$. If $(x_1, x_2) \in (X\times X)\setminus A$, then $f(x_1) \neq f(x_2)$ .Since $(Y,\delta^*(J))$ is fibrewise ideal T_2-space , there exists disjoint I-open sets V_1, and V_2 such that $f(x_j) \in V_j$ for $j = 1,2$, then by I-continuity of f , Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in Io(x_1, x_2)$, since the product of two open sets is open set . Therefore $f^{-1}(V_1) \times f^{-1}(V_2) \subset (X\times X)\setminus A$. It follows that $(X\times X)\setminus A$ is I-open , and hence A is I-closed set in $X\times X$. ■

Theorem : 4-4-7

Let $(X , \tau^*(I))$ and $(Y,\delta^*(J))$ be two fibrewise ideal topological spaces over B, and $f : (X , \tau^*(I)) \to (Y,\delta^*(J))$ is injective, surjective and I-open, then $(Y,\delta^*(J))$ is fibrewise ideal T_2-space over B if $(X ,\tau^*(I))$ is fibrewise ideal T_2-space over B.

Proof:

Let $y_1, y_2 \in Y_b$ such that $y_1 \neq y_2$. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are different points of X_b . Since f is surjective there exists $x_1, x_2 \in X_b$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ from hypothesis $(X ,\tau^*(I))$ is a fibrewise ideal T_2-space , so there exists $U,V \in \tau^*$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. This implies that $f(x_1) = y_1 \in f(U)$, $f(x_2) = y_2 \in f(V)$. Since f is I-open , then $f(U), f(V) \in \delta^*$, and f is injective, $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Thus $(Y,\delta^*(J))$ is fibrewise ideal T_2-space over B.■

Theorem: 4-4-8

In fibrewise ideal T_2-space over B a sequence converges to unique point.

Proof:

Assoming that x and y are two distncet points and (x_n) converges to x and y . Since $(X, \tau^*(I))$ is fibrewise ideal T_2-space over B, there exists U, $V \in \tau^*$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since (x_n) converges to x and U is neighborhood of x , then there exist $n_1 \in N$ such that $x_n \in U$ for all $n \geq n_1$ .Since (x_n) converges to y and V is a neighborhood of y , then there exist $n_2 \in N$ such that $x_n \in V$ for all $n \geq n_2$, let $n_0 = \max\{n_1, n_2\}$

then for all $n \geq n_0$, $x_n \in U$ and $x_n \in V$. Hence $U \cap V \neq \emptyset$. This is a contradiction. ■

Theorem: 4-4-9

Let A be a compact set in a fibrewise ideal T_2-space $(X, \tau^*(I))$ over B, then A is I-closed .

Proof:

Let $x \in A^c$. For each $y \in A$, we have $x \neq y$. So there are disjoint I-open sets U and V. So that $x \in U$ and $y \in V$. Then $\{V: y \in A\}$ is an I-open cover of A . Let $\{V_1, V_2, \ldots V_n\}$ be a finite subcover . Then $\bigcap_{i=1}^{n} V_i$ is an I-open set containing x and contained in A^c. Thus A^c is I-open set and A is I-closed set .■

4-5 Higher separation axioms in fibrewise ideal topological spaces:

The aim of this section is to study higher separation axioms in fibrewise ideal topological spaces over B.

Definition: 4-5-1

A fibrewise ideal topological space $(X, \tau^*(I))$ over B is said to be fibrewise ideal regular, if every non- trivial fibrewise subspace is regular.

Definition: 4-5-2

A fibrewise ideal topological space $(X, \tau^*(I))$ over B is said to be a fibrewise ideal T_3-space if every non- trivial fibre subspace is regular and T_1-space .

Example: 4-5-3

Let X = Ƶ is the set of integers , τ_1 is the discrete topology on Ƶ , B = Ƶ is the set of integers numbers τ_2 is the discrete topology , I ={∅} is fibrewise ideal on X.

p : (X ,τ_1) → (B ,τ_2) defined by p(x) = $\begin{cases} 1 & \text{if } x \in Ƶ^+ \\ 0 & \text{if } x \notin Ƶ^+ \end{cases}$

then p is continuous and for any b ∈ B

$$X_b = p^{-1}(b) = \begin{cases} Ƶ^+ & \text{if } b = 1 \\ Ƶ^- \cup \{0\} & \text{if } b = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Where $Ƶ^+$ = {1,2,……} , $Ƶ^-$ = {……..,-2,-1}

So every non-trivial fibre subspace is discrete and so $(Ƶ,\tau^*(I))$ is fibrewise ideal T_3- topological space over B .

Theorem: 4-5-4

In a fibrewise ideal topological space $(X,\tau^*(I))$ over B, X is fibrewise ideal regular iff for every I-open set V containing x∈ X , there exists an I-open set U of X such that x∈ U ⊂ $cl^*(U)$ ⊂ V.

Proof:

⟹: let V be an I-open subset such that $x \in V$. Then X\V is an I-closed set not containing x. Therefore there exists disjoint I-open sets U and W such that $x \in U$ and $X \setminus V \subset W$. Now $X \setminus V \subseteq int(W)$, implies $X \setminus int(W) \subset V$. Again $U \cap W = \emptyset$, implies $U \cap int(W) = \emptyset$, which implies that $cl^*(U) \subset X \setminus int(W) \subset V$. Therefore $x \in U \subset cl^*(U) \subset V$.

⟸ let F be an I-closed set not containing x. By hypothesis, there exists an I-open set U such that $x \in U \subset cl^*(U) \subset X \setminus F$. If $W = X \setminus cl^*(U)$, then U and W are disjoint I-open sets such that $x \in U$ and $F \subset W$. ■

Definition: 4-6-1

A fibrewise ideal topological space $(X, \tau^*(I))$ over B is said to be a fibrewise ideal completely regular , if every non- trivial fibre subspace is completely regular i-e for every I-closed set $A \subset X_b$ and any $x \notin A$, there exist a continuous function $f : X_b \to I$ such that $f(x) = 0$, $f(A) = 1$, where I is the unit interval , $b \in B$.

Theorem :4-6-2

Every subspace of fibrewise ideal completely regular $(X, \tau^*(I))$ is a fibrewise ideal completely regular.

Proof:

Let $(X, \tau^*(I))$ be a fibrewise ideal completely regular space over B , and Y be a subset of X , let $x \in Y$ and V be I-closed set in Y not containing x . Then V = A∩Y, where A is an I-closed set in X, Hence $x \in Y \subseteq X$, implies $x \in X$ and A is a closed set in X not containing x , since X is fibrewise ideal completely regular over B , then there is continuous function $f: X \rightarrow I$ such that $f(x) = 0$, $f(A) = 1$.

Let $g = f/_Y$ then $g : Y \rightarrow I$ is continuous since $V \subseteq A$, implies $g(V) = 1$, $g(x) = 0$. Therefore $(Y, \tau_Y^*(I_Y))$ is a fibrewise ideal completely regular over B . ■

Example: 4-6-3

Let X = B = ℝ and τ_1, τ_2 are the usual topology on ℝ . If $P : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_2)$ defined by $p(x) = x$, and let I = { A⊆ ℝ: A is a finite subset of ℝ }. Then $(\mathbb{R}, \tau_1)$ fibrewise topological space over $(\mathbb{R}, \tau_2)$ and $(\mathbb{R}, \tau^*(I))$ is the usual topology , and hence $(\mathbb{R}, \tau_1^*(I))$ is fibrewise completely regular .

Theorem : 4-6-4

If $\phi : (X, \tau_1) \rightarrow (Y, \tau^*(I))$ is a fibrewise continuous bijective and open function and X is fibrewise completely regular over B , then Y is a fibrewise ideal completely regular over B.

Proof :

Let y be any point in Y, H is I-closed set such that $y \notin H$, since ϕ is surjective continuous function then there exist x in X such that

$y = \phi(x)$, and $\phi^{-1}(H) = A$ is I-closed set in X , and $x \notin \phi^{-1}(H)$. Since X is fibrewise completely regular, there is a continuous function $f : X \to I$ such that $f(x) = 0$, $f(A) = 1$, the composition function $f \circ \phi^{-1} = q : Y \to I$ is continuous function such that $q(y) = 0$, $q(H) = 1$. Thus Y is a fibrewise ideal completely regular over B .■

Definition: 4-7-1

A fibrewise $(X,\tau^{*}(I))$ space over B is called fibrewise ideal Tychonoff space (or $T_{3\frac{1}{2}}$) if it is fibrewise ideal completely regular , T_1- space over B.

Theorem: 4-7-2

If $(X, \tau^{*}(I))$ is a fibrewise ideal $T_{3\frac{1}{2}}$ space over B , then $(X , \tau^{*}(I))$ is a fibrewise ideal T_3-space over B .

Proof:

Suppose F is I-closed set in X not containing x . If X is fibrewise ideal $T_{3\frac{1}{2}}$, we can choose any continuous function with $f(x) = 0$ and $f(F) = 1$. Then $U = f^{-1}(\infty, \frac{1}{2})$ and $V = f^{-1}(\frac{1}{2}, \infty)$ are disjoint I-open sets with $x \in U$, $F \subseteq V$ .Therefore X is fibrewise ideal regular .Since X is fibrewise ideal T_1 . Then X is T_3. ■

Theorem :4-7-3

If ($X_{\alpha}, \tau^{*}_{\alpha}(I)$) is a fibrewise ideal topological spaces over B for all $\alpha \in \Lambda$. Then X = $\prod_{\alpha\in\Lambda} X_\alpha$ is a fibrewise ideal $T_{3\frac{1}{2}}$ space over B iff each ($X_{\alpha}, \tau^{*}(I_\alpha)$) is a fibrewise ideal $T_{3\frac{1}{2}}$ space over B.

Proof:

⟹: If X =$\prod_{\alpha\in\Lambda} X_\alpha$ is fibrewise ideal $T_{3\frac{1}{2}}$ then each X_α is homeomorphic to a subspace of X , so each X_α is fibrewise ideal $T_{3\frac{1}{2}}$ space .

⟸ : Conversely , suppose each X_α is fibrewise ideal $T_{3\frac{1}{2}}$ space over B , and that F is an I-closed set in X not containing a.There is a basic I-open set U such that a∈ U = $\bigcap_{i=1}^{n} \pi_{\alpha i}^{-1}(U_i)$ ⊆ X\F . For each i we can bick a continuous function f_{α_i} : $X_{\alpha_i} \rightarrow$ [0,1] with $f_{\alpha_i}(a_{\alpha_i}) = 0$, and $f_{\alpha_i}(X\backslash U_i) = 1$. Define f: X→ [0,1] by $f(x) = \max\{ f_{\alpha_i} \circ \Pi_{\alpha_i})(x)\}_{i=1}^{n}$.
Then f is continuous and $f(a) = \max \{ f_{\alpha_i}(a_{\alpha_i})\}_{i=1}^{n} = 0$

If x∈ F, then for some i , $x_{\alpha i} \notin U_{\alpha i}$, and $f_{\alpha_i}(x_{\alpha_i}) = 1$, so $f(x) = 1$. Therefore $f(F) = 1$ and X is fibrewise ideal $T_{3\frac{1}{2}}$ space over B .■

Definition: 4-8-1

A fibrewise ideal topological space (X,$\tau^{*}(I)$) over B is said to be fibrewise ideal normal if every non- trivial fibrewise subspace is normal space over B.

Definition: 4-8-2

A fibrewise ideal topological space $(X,\tau^*(I))$ over B is said to be a fibrewise ideal T_4-space over B if it is fibrewise ideal normal space as well as a fibrewise ideal T_1-space over B.

Example: 4-8-3

Let X = B = $\mathbb{R}$ and τ_1 , τ_2 are the trivial topologies on $\mathbb{R}$, and I $=\{\phi\}$ is an ideal on X , P : $(X , \tau_1) \rightarrow (B, \tau_2)$ defined by p(x) = x then X is fibrewise topological space over B, and A^*= cl(A) , then $cl^*(A) = A\cup A^*$ = cl(A) for any set $A\subset \mathbb{R}$. So $\tau^* = \tau$. Hence every non-trivial fibre subspace is a fibrewise ideal T_4-space. So $(\mathbb{R} , \tau^*(I))$ is a fibrewise ideal T_4-space over B.

Theorem: 4-8-4

Let $(X,\tau^*(I))$ be a fibrewise ideal topological space over B, where I is completely codense, and if for any disjoint I-closed sets A and B, there exists disjoint I-open sets U and V such that $A\subseteq U$ and $B\subseteq V$. Then for any I-closed set A and I-open set U containing A, there exists an I-open set U such that $A\subset U\subseteq cl^*(U) \subseteq V$.

Proof:

Suppose A is I-closed and V is I-open set containing A. Since A and X\V are disjoint I-closed sets , there exists disjoint I-open sets U and

W such that A⊆ U and X\V ⊆W, Since X\V is I-closed , and W is I-open , X\V ⊆ int^*(W). Then X\ int^*(W)⊆V. Again U∩W = ϕ ,then U ∩ int^*(W) = ϕ , U ⊆ X \ int^*(W). Then cl^*(U) ⊆ X \ int^*(W) ⊆ V. Thus U is the required I-open set with A⊆ U⊆ cl^*(U) ⊆ V .■

Theorem: 4-8-5

Let (X,τ^*(I)) be a fibrewise ideal normal space over B, then every closed fibrewise ideal subspace of (X, τ^*(I)) is fibrewise ideal normal space over B.

Proof:

Let (X,τ^*(I)) be a fibrewise ideal normal space over B, and Y be a closed subspace of X. To prove (Y, $\tau^*_Y(I_Y)$) is fibrewise normal space over B, with the relative topology. Let H and K be two I-closed disjoint subsets of Y. Then we have H = Y∩ A, K = Y∩ B, where A and B are I-closed sets in X . Now Y is I-closed and A and B are I-closed.

Hence Y∩ A and Y∩ B are disjoint I-closed subsets of X. Since (X,τ^*(I)) is fibrewise normal corresponding to the disjoint I-closed subsets H and K of X , there exists I-open subsets U and V such that H⊂ U, K⊂ V, U∩ V = ϕ . Now H⊂ U, H⊂ Y so H⊂ U∩ Y, K⊂ V, K⊂ Y, Hence K⊂ V∩ Y. Also U∩ V = ϕ. Therefore (Y∩U) ∩ (Y∩V) = ϕ . Since U and V are I-open sets in X , and hence Y∩U and Y∩V are I-open sets in Y . Now corresponds to the two I-closed sets H and K of Y , there exists I-open set Y∩U and Y∩V in Y such that H⊂Y∩U, K⊂Y∩V, and (Y∩U) ∩ (Y∩V) = ϕ. Hence (Y, $\tau^*_Y(I_Y)$) is fibrewise normal space over B .■

Theorem :4-8-6

Fibrewise normality invariant under continuous I-closed surjective map.

Proof:

Let (X, τ^*(I)) be an ideal fibrewise normal space over B , and Y is a fibrewise ideal topological space over B ,and let f:(X,τ^*(I))→(Y,τ^*(J)) be a fibrewise continuous , I-closed and surjective. To prove (Y, τ^*(J)) is fibrewise normal space over B. Let F_1, and F_2 be disjoint closed sets in Y. Since f is continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are I-closed in X. Since $F_1 \cap F_2 = \phi$.This implies that $f^{-1}(F_1) \cap f^{-1}(F_2)$ =. Now X is fibrewise ideal normal space and $f^{-1}(F_1)$, $f^{-1}(F_2)$ are disjoint I-closed subsets in X.Hence there exists I-open sets U and V such that $f^{-1}(F_1) \subseteq U$, $f^{-1}(F_2) \subseteq V$ and $U \cap V = \phi$. put $W_1 = Y \setminus f(X\setminus U)$. Since f is I-closed map and X\U is I-closed f(X\U) is I-closed set in Y. Hence W_1 is I-open in Y. Also $f^{-1}(F_1) \subseteq U$, $X\setminus U \subseteq X\setminus f^{-1}(F_1)$, so $X\setminus U \subseteq f^{-1}(Y\setminus F_1)$. So $f^{-1}(W_1)$ = $f^{-1}(Y\setminus f(X\setminus U) = X\setminus f^{-1}(f(X\setminus U) \subseteq X\setminus(X\setminus U) = U$.

Hence $f^{-1}(W) \subseteq U$. Thus there exists I-open set W_1 containing F_1, such that $f^{-1}(W_1) \subseteq U$. Similarly there exists W_2 such that $f^{-1}(W_2) \subseteq V$.

$f^{-1}(W_1) \cap f^{-1}(W_2) \subseteq U \cap V = \phi$. Thus $f^{-1}(W_1 \cap W_2) = \phi$ and so $W_1 \cap W_2 = \phi$. Thus there exists I-open sets W_1 containing F_1 and W_2 containing F_2 such that $W_1 \cap W_2 = \phi$. Hence $(Y, \tau^*(J))$ is fibrewise ideal normal space over B.■

Definition: 4-9-1

A fibrewise ideal topological space $(X, \tau^*(I))$ over B is said to be fibrewise completely normal iff every non-trivial fibre subspace is completely normal, that is for any two separated sets A and C of X_b, where $b \in B$, there exists disjoint I-open sets G and H in X_b such that $A \subset G$, $C \subset H$.

Definition: 4-9-2

A fibrewise ideal topological space $(X, \tau^*(I))$ over B is said to be fibrewise ideal T_5-space over B if it is a fibrewise ideal completely normal as well as fibrewise ideal T_1-space over B.

Example: 4-9-3

Let $X = \mathbb{R}$, τ_1 is the discrete topology on $\mathbb{R}$, and $B = \mathbb{R}$, τ_2 the trivial topology on $\mathbb{R}$ and I = {A: A is finite subset of [0,1] } is an ideal on X .

$$A^*(I,\tau)=\begin{cases}\bar{A}, & \text{if } A \text{ is not finite subset of } [0,1]\\ \emptyset & \text{if } A \text{ is finite subset of } [0,1]\end{cases}$$

$cl^*(A) = A\cup A^* = \bar{A}$ then $\tau^* = \tau_1$

If $P : (X , \tau_1) \rightarrow (B , \tau_2)$ defined by $p(x) = x$ then P is continuous and $X_b = p^{-1}(b) = \{b\}$ is a trivial fibrewise topological space over B.

So $(\mathbb{R}, \tau^*(I))$ is a fibrewise ideal T_5- space over B.

Theorem: 4-9-4

i) Every fibrewise ideal completely normal topological space over B is fibrewise ideal normal topological space over B.

ii) Every fibrewise ideal T_5- topological space over B is fibrewise ideal T_4- topological space over B.

Proof:

i) Let A and C be two disjoint I-closed subsets of X_b, where $b\in B$. Therefore $A = cl(A)$ and $C = cl(C)$ and $A\cap C = \phi$. Which implies $cl(A) \cap C = A\cap C = \phi$, and $A\cap cl(C) = A\cap C = \phi$, therefore A and C are separated sets .Since $(X,\tau^*(I))$ is a fibrewise ideal completely normal space over B, there exists I-open sets G and H such that $A\subset G$ and $C\subset H$, and $G\cap H = \phi$, Hence $(X , \tau^*(I))$ is fibrewise ideal normal space over B.

ii) Let (X, τ^*(I)) be fibrewise ideal T_5- space over B, then (X, τ^*(I)) is fibrewise completely normal as well as fibrewise T_1- space over B. By **(i)**, hence (X, τ^*(I)) is fibrewise normal as well as fibrewise T_1- space over B. Therefore (X,τ^*(I)) is fibrewise ideal T_4- space over B.■

Theorem: 4-9-5

Every subspace of fibrewise ideal completely normal topological space over B is fibrewise ideal completely normal topological space over B.

Proof:

Let (X, τ^*(I)) be a fibrewise ideal completely normal space over B and Y be a subspace of X. To prove (Y, τ^*(I)) is fibrewise ideal completely normal with the relative topology. Let A and C be separated sets in Y. Then we have $cl_Y(A) \cap C = \phi$, $A \cap cl_Y(C) = \phi$. Now $cl_Y(A) = cl(A) \cap Y$, $cl_Y(C) = cl(C) \cap Y$.

Then $\phi = A \cap cl_Y(C) = A \cap (cl(C) \cap Y) = (A \cap cl(C)) \cap Y = A \cap cl(C)$ Since $A \cap cl_Y(C) \subset A \subset Y$. Similarly, $\phi = cl_Y(A) \cap C$. Which implies that A and C are separated sets in X, and since X is fibrewise ideal completely normal topological space over B, there exists I-open sets G and H in X such that $A \subset G$, $C \subset H$, and $G \cap H = \phi$.

Now A$\subset$ G and A$\subset$ Y, so A$\subset$ G $\cap$Y, let G $\cap$Y = U , then A$\subset$ U , and C $\subset$ Y, so C $\subset$ H $\cap$ Y, let H$\cap$Y = V , then C $\subset$ V , where U,V I-open sets in Y, U$\cap$V = (G$\cap$ Y) $\cap$(H$\cap$ Y) = (G$\cap$ H) $\cap$ Y= ϕ.

Hence (Y, τ_Y^*(I)) is fibrewise ideal completely normal space over B.■

Conclusion

Given a topological spaces (X , τ), let (X , $\tau^*(I)$) be a fibrewise ideal topological space over B with a projection P: X $\rightarrow$ B and fiber subspaces { X_b : b$\in$ B } . We called a fiber subspace X_b is trivial if X_b = $\emptyset$ or X_b = X and we defined a fibrewise ideal topological space to be T_i-space if every non-trivial fibre subspace is T_i -space for i=0,1,2,3,4,5 ; where T_i is the separation axiom for all i . During our study we found that :

1/ If (X , τ) is a T_i-space, then (X, $\tau^*(I)$) is a fibrewise ideal T_i topological space over B for i = 0,1,2.

2/ If (X , $\tau^*(I)$) is a T_i-space, then (X , $\tau^*(I)$) is a fibrewise ideal T_i topological space over B for i = 0,1,2,3,4,5 .

3/ (X , $\tau^*(I)$) is a fibrewise ideal T_i topological space over B without being(X , $\tau^*(I)$) is a T_i-space for i = 0,1,2,3,4,5.

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ملخص

إذا كان (X, τ) فضاء تبولوجي، I مثالي على X و $\tau^*(I)$ التبولوجيا الأقوى من التبولوجيا τ المولدة بواسطة المثالي I ، فإن لأي فضاء تبولوجي B بحيث تكون $P : X \rightarrow B$ دالة مستمرة يسمى الفضاء $(X, \tau^*(I))$ فضاء مثالي ليفي على B وله فضاءات جزئية ليفية

$\{X_b : b \in B\}$ حيث $X_b = P^{-1}(b)$ لكل $b \ni B$.

الهدف من الرسالة هو تعريف مسلمات الفصل في الفضاءات المثالية الليفية ودراسة بعض الخواص الأساسية . أيضاً سوف نناقش المفاهيم الأساسية والنتائج المهمة في هذا الموضوع بما في ذلك العلاقة بين هذه المسلمات وعلاقتها بمسلمات الفصل المعروفة.