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فضاءات هيلبرت

Hilbert Spaces

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Dedication

The researcher dedication this work to

My parents, husband, brothers, sisters, friends and any other persons who gave me a hand to achieve this work.

Abstract

We present studying on Hilbert space and linear operators.

It has been studied some of the fundamental concepts of inner product spaces. Some of these concepts are orthogonal and orthonormal sets that play important role in constructing Hilbert spaces. As Hilbert space have been defined and supported with some examples upon them. Some fundamental theorems are also presented that are in relation to these spaces. Such as Bessel inequality, Gram Schmidt process in inner product space. And Riez`s Theorem.

The researcher has introduced the properties of the linear operators,,linear functional ,selif-adjoint linear operators and their influences on Hilbert spaces, which are very important in functional analysis.

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Notations

$B[a,b]$ Space of bounded functions

$B[X,Y]$ Space of bounded linear operators

c A sequence space

$\mathbb{C}$ Field of complex numbers

C^n Unitary n-space

$C[a,b]$ Space of continuous functions

$D(T)$ Domain of an operator

$d(x,y)$ Distance from x to y

$\dim X$ Dimension of a space X

$\|f\|$ Norm of bounded linear functional f

$L^p[a,b]$ A function space

l^p A sequence space of l^p

l^∞ A sequence space of l^∞

$L[X,Y]$ A space of linear operators

$N(T)$ Null space of an operator

$\mathbb{R}$ The field of real numbers

$\mathbb{R}^n$ Euclidean n-space

$span\ M$ Span of a set M

T^* Hilbert-adjoint operator of T

X^* X dual space of a vector space

$\|z\|$ Norm z

$\langle y, z \rangle$ Inner product of y and z

$y \perp z$ y is orthogonal to z

$X^{\perp}$ Orthogonal complement of a closed subspace X

Introduction

Functional analysis is an abstract branch of mathematical science. It studies functions of spaces and involves vector spaces of any dimension[2]. It also studies the operators that are defined on the vector spaces [10] . Also it includes study of transforms such as Fourier transforms which they how some applications in differential and integral equations . In addition, it studies the sequences defined on functions spaces[16]. This study aims to study Hilbert spaces and some of their applications. Moreover , it aims to study linear operators ,linear functionals and their applications on Hilbert spaces.

Hilbert spaces due to the German Mathematician **David Hilbert**(1862 -1943). The study of these spaces were introduced in the axioms of Newman`s work [9]. Hilbert spaces play an important role in partial differential equations theorems , Quantum mechanics ,Fourier transforms and their applications [6].

In the first chapter , it has been studied some principle concepts and examples that with Hilbert spaces, such as metric spaces, vector spaces, sequences, normed spaces, the bounded linear operators and the linear functionals.

In the second chapter , it has been studied inner product space, Hilbert spaces, orthogonal, orthonormal. Some theorems that are related to them. Also some properties of the inner product, direct sum and orthogonal complement .

In the third chapter, it has been studied the linear functionals on Hilbert spaces, the sesquilinear functional, Hilbert-Adjoint operator, some examples and theorems that are related to them.

Chapter One

1 Some Fundamental Concepts

This chapter aims to introduce some principle concepts, which have great importance in studying Hilbert spaces, such as metric spaces ,normed spaces which are defined on vector spaces. So that it is so essential to show vector spaces and know their properties geometrically. We will be showed some principle definitions.

1.1 Metric Spaces

Metric spaces can be considered as a basic spaces. The ideas of convergence and continuity exist. The fundamental ingredient that is needed to make these concepts is a distance, also called a metric, which is a measure of how elements close to each other [15].

Definition (1.1.1)

A distance (or metric) on a non-empty set X is a function.

$$d : X \times X \longrightarrow \mathbb{R}^{+} \cup \{0\}$$

$$(x, y) \longmapsto d(x, y)$$

Such that the following properties (called axioms) hold for all $x, y, z \in X$,

1) $d(x,y) \leq d(x,z) + d(z,y)$, $(Triangle\ inequality)$,
2) $d(y,x) = d(x,y)$, $(Symmetry)$
3) $d(x,y) \geq 0\ \forall\, x,y \in X\ \ and\ \ d(x,y) = 0\ \ if\ and\ only\ if\ x = y,$

The pair (X, d) is **called Metric Space .**

In stead of (X, d) we may simply write X.

1.2 Normed Spaces

" If we take a vector space and define a metric on it using a norm, we can obtain the metric spaces. A normed space is the name given to the resulting area. It is then referred to as a Banach space if it is a full metric space. They are the developed of functional analysis, and on them are defined Banach spaces of linear operators. The fundamental concepts of these theories are presented in this chapter"[13].

Vector space plays role in many branches of mathematics . A vector space is Hilbert space (linear space). Additionally, this section includes background information on these spaces. [19].

Definition (1.2.1)

If X is a nonempty set of elements x, y, z,... and F is a field of scalars,,..., then x+y in X and x in X correspond to a third element, known as the scalar product of and x, such that addition and multiplication meet the following criteria.

1) $x + y = y + x \quad \forall\, x, y \in X$

2) $x + (y + z) = (x + y) + z \quad \forall\, x, y, z \in X$,

3) there is a unique element 0in X, called zero element , such that $x + 0 = x$, $\forall x \in X$,

4) $\forall x \in X$, there is a unique element $(-x)$ in X such that $x + (-x) = 0$,

$a)$ $\alpha(x + y) = \alpha x + \alpha y \quad \forall\, x, y \in X, \alpha \in F$,

$b)$ $(\alpha\beta)x = \alpha(\beta x) \quad \forall\, x \in X, \alpha, \beta \in F$ *and*

$c)$ $1x = x \ \forall\, x \in X$, where $1 \in$ F is the identity in F.

Then $(X, +, .)$ Satisfying properties $((1) - (4))$ *and* $((a) - (c))$ referred to as a vector space over F. The components of X are known as vectors or points, while the components of F are known as scalars. A complex vector space is (X,+,.) if F is the field of complex numbers C [resp - real numberR] [14]

Definition(1. 2.2)

A subspace of a vector space X is a nonempty subset Y of X such that we have $\alpha y_1 + \beta y_2 \in Y$ for every $y_1, y_2 \in Y$ and all scalars α, β. Y is a vector space in and of itself. These two algebraic operations are those that X induces.

Definition (1.2.3)

It is argued that a finitely many-vector series $\{x_1, x_2, \dots, x_n\}$is linearly independent if the relation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n = 0$$

Holds in case when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$; otherwise , the of elements sequence $x_1, x_2, \dots, x_n$ is said to be linear dependent .

Definition (1.2.4)

A basis is a collection of linearly independent vectors with the property that each vector x∈X can be a linear combination of some subset of B if X is a vector space and B is a collection of linearly independent vectors.

Definition (1.2.5)

If there is a positive integer n such that X includes a linearly independent collection of n vectors, then the dimension of the vector space X is finite. Any collection of n+1 or more X vectors is linearly dependent, and n is referred to as the X dimension, denoted by the formula n=dimX.

X=0 has a finite number of dimensions, and dimX=0

Let X have infinite dimensions rather than finite ones.

Definition (1.2.6)

A vector space with a norm defined on which is called a Normed space (X). A complete normed space is a banach space. Here, a vector space norm (real or complex) A positive real-valued function on X is called X, and its value at x∈X is represented by$\|\cdot\|: \quad z \longrightarrow \mathbb{R}^+ \cup \{0\}$

1) $\|x\| \geq 0 \quad \forall\, x \in X$
2) $\|x\| = 0 \quad if\ and\ only\ if \quad x = 0$
3) $\|\alpha x\| = |\alpha|\|x\| \ \forall\, x \in X, \alpha \in F\ , (F = \mathbb{R}\ or\ \mathbb{C})$
4) $\|x + y\| \leq \|x\| + \|y\| \quad (Triangle\ inequality) \forall x, y \in X$

with the aforementioned traits

A metric d on X defined by d(x, y)= $\|x - y\|$, (x ,y ∈X), also known as the metric by the norm, is said to be the metric on X.

(X, ||·||) or just X serves as the definition of the normed spaces.

We will see later in this part that not all of the metrics on a vector space can be derived from a norm, as was mentioned in earlier sections where some of the metric spaces may be converted into normed spaces [12].

Next we give some examples.

Examples(1.2.1)

1)If $X = \mathbb{R}^n$, and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ such that $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$,then

$\|x\| = (\sum_{i=1}^{n} x_i{}^2)^{\frac{1}{2}}$ for all $x \in \mathbb{R}^n$ defin norm on $\mathbb{R}^n$, hence $(\mathbb{R}^n, \|\cdot\|)$is a normed space.

2)If $X = l^p$, such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ $(p \geq 1$,fixed) , In the space l^p, each element is a sequence.$X = (x_i) = (x_1, x_2, \dots)$ of numbers ,then

$\|x\| = (\sum_{i=1}^{\infty} \|x_i\|^p)^{\frac{1}{p}}$ for all $x_i \in l^p$ define a norm on l^p and given by

$$d(x, y) = \|x - y\| = \left(\sum_{j=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

3)If $X = \mathbb{C}^n$, then

$\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ for all $x \in \mathbb{C}^n$

define norm on $\mathbb{C}^n$, that is $(\mathbb{C}^n, \|\cdot\|)$ is a normed space.

Definition (1.2.7)

Suppose X is a metric space. A sequence of points$\{x_n\}_{n \in N}$ converges to the point in X $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0 .$$

That is for every $\epsilon > 0$ there must exist some integer $N > 0$ such that

$$d(x_n, x) \leq \epsilon \ \forall\ n \geq N.$$

In this case, we write $x_n \longrightarrow x$.

Examples(1.2.2)

In any metric space $x_n \longrightarrow x$ if and only if $d(x_n, x) \longrightarrow 0 \; as \; n \longrightarrow \infty$ (because $x_n \in B_\varepsilon(x)$ if and only if $d(x_n, x) < \varepsilon$) . for example, $x_n \longrightarrow x$ when $d(x_n, x) \leq \frac{1}{n}$ hold .

Definition (1.2.8)

If X is a metric space and for every $\epsilon > 0$ there exists an integer $N > 0$ .A sequence of points$\{x_n\}_{n\in N}$ in X is a Cauchy sequence like that

$d(x_m, x_n) < \varepsilon \quad \forall\, m, n \geq N$.

Definition (1.2.9)

A series converges is a sequence of vectors in a normed space obtained by addition , $(x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots)$; the sequence's N^{th} term is denoted by $S_n = \sum_{n=1}^{N} x_n$, $N \in \mathbb{N}$ (The sequence partial sums).

Therefore, the series $\sum_n x_n$ **is convergent** to x if $\|x - \sum_{n=1}^{N} x_n\| \longrightarrow 0$ when $N \longrightarrow \infty$.

In this case the limit x is called its sum

$$x_1 + x_2 + \cdots = \sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n = x.$$

A series is called the **converge absolutely** when $\sum_n \|x_n\|$ converges in $\mathbb{R}$.

Definition(1.2.10)

If every Cauchy sequence in a metric space (X,d) converges to a point in X, then the space is said to be complete.

Definition (1.2.11)

Let there are two metric spaces, (X,d) and $(\grave{X}, \grave{d})$. If $d\langle T(y), T(z)\rangle = d\langle y, z\rangle$ for any y ,z ∈X, a mapping T from X to $\grave{X}$ is an isometry.

Definition (1.2.12)

If X a metric space is called the **separable** if it contains a countable dense sub set A ,where A is countable and $\overline{A} = X$.

Therefore, since subspace Y of a Banach space X is a subspace of X taken into account as a normed space, we do not require Y to be complete.

Theorem (1.2.1) [13]

If the space $(X, \|\cdot\|)$is normed. Following that, a dense in $\hat{X}$ Banach space $\hat{X}$ and an isometry A form X onto a subspace W of $\hat{X}$ are present. With the exception of isomorphism, the space $\hat{X}$is unique.

Proof

Since a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ and an isometry $A: X \longrightarrow W = A(X)$, where W is dense in $\hat{X}$ is unique , except for isometries. We must first turn $\hat{X}$ into a vector space before imposing an appropriate norm on it. We consider any $\hat{x}, \hat{y} \in \hat{X}$ in order to define on $\hat{X}$ the two algebraic operations of a vector space. and representatives $(x_n) \in \hat{x}$ and $(y_n) \in \hat{y}$. Since the equivalence classes of Cauchy sequences in X are $\hat{x}\, and\ \hat{y}$. $z_n$is set to be equal to $x_n + y_n$. Therefore, $(z_n)$is Cauchy in X because

$$\|z_n - z_m\| = \|x_n + y_n - (x_m + y_m)\| \leq \|x_n - x_m\| + \|y_n - y_m\|.$$

We define the equivalence class for which (z_n) is a representative as the sum $\hat{z} = \hat{x} + \hat{y}$ of $\hat{x}$, $\hat{y}$; hence, $(z_n) \in \hat{z}$. This concept is not dependent on the Cauchy sequences chosen to represent $\hat{x}$ and $\hat{y}$. since if $(x_n) \sim (\acute{x}_n)$ and $(y_n) \sim (\acute{y}_n)$, then $(x_n + y_n) \sim (\acute{x}_n + \acute{y}_n)$ because $\alpha \acute{x} \in \acute{X}$ which (αx_n)

$$\|x_n + y_n - (\acute{x}_n + \acute{y}_n)\| \leq \|x_n - \acute{x}_n\| + \|y_n - \acute{y}_n\|.$$

We have defined the equivalence for which (αx_n) is a representative as the product $\alpha \acute{x} \in \acute{X}$of a scalar α and $\acute{x}$. The selection of an $\acute{x}$ representative has no bearing on this definition. The equivalence class containing all Cauchy sequences that converge to zero is represented by the zero element of $\acute{X}$. As a result, $\acute{X}$ is a vector space. According to the definition, the vector space operations induced from $\acute{X}$ and those induced from X using A agree on W. A creates a norm $\|\cdot\|_1$ on W whose value at each of the points $\hat{y} = Ax \in W$ is $\|\hat{y}\|_1 = \|x\|$. Given that A is isometric, the restriction of $\hat{d}$ to W is the equivalent metric on W. By going beyond the norm $\|\cdot\|_1$to $\acute{X}$ by setting $\|\hat{x}\|_2 = \acute{d}(\acute{0}, \acute{x})$for every $\acute{x} \in \acute{X}$.

1.3 Finite Dimensional Normed Spaces and Subspaces.

" Due to the significant role that finite dimensional normed spaces and subspaces play in Hilbert space. We are unable to find a linear combination that contains large scalars but represents a small vector in the case of linear independence of vectors"[3]

Lemma (1.3.1) [13]

If a normed space X (of any dimension) contains a collection of linearly independent vectors named $\{x_1, x_2, \dots, x_n\}$. Then there is an integer c>0 such that for each of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$we obtain

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \ ;(\ c > 0) \longrightarrow (1)$$

Proof

We write $s = |\alpha_1| + \cdots + |\alpha_n|$ .If $s = 0$,then all α_j are zero for all $1 \leq j \leq n$, Therefore, (1) is true for each c

Let s>0 . If $\beta_j = \alpha_j/s$, then (1) is similar to the inequality that we derive from (1) by multiplying by s .Thus

$$\|\beta_1 x_1 + \cdots + \beta_n x_n\| \geq c \qquad ; \ \left(\sum_{j=1}^{n} |\beta_j| = 1\right) \ \longrightarrow (2)$$

for each n-tuple of scalars $\beta_1, \dots, \beta_n$ with $\sum_{j=1}^{n} |\beta_j| = 1$,since(2) holds.

Let's say that is false. Then a sequence (y_m) of vectors

$$y_m = \beta_1{}^{(m)} x_1 + \cdots + \beta_n{}^{(m)} x_n \quad \left(\sum_{j=1}^{n} \left|\beta_j{}^{(m)}\right| = 1\right)\text{exists}$$

Such that $\beta_1, \dots, \beta_n$ with $\sum_{j=1}^{n} |\beta_j| = 1$

$$\|y_m\| \longrightarrow 0 \ \ as\ m \longrightarrow \infty.$$

Since $\sum\left|\beta_j{}^{(m)}\right| = 1$, hence $\left|\beta_j{}^{(m)}\right| \leq 1$. So that for any fixed j the sequence

$$\left(\beta_j{}^{(m)}\right) = \left(\beta_j{}^{(1)}, \beta_j{}^{(2)}, \dots\right)$$

is bounded . Since $\left(\beta_1{}^{(m)}\right)$ has a convergent subsequence. If β_1 denote the limit of that subsequence, if (y_1, m) the corresponding subsequence of (y_m) .Also (y_1, m) has a subsequence $(y_2, m)$for which the corresponding subsequence of scalars $\beta_2{}^{(m)}$ converges ; if β_2 denote the limit .Continuing in this way, after n steps we obtain a subsequence $\left(y_{n,m}\right) = \left(y_{n,1}, y_{n,2}, \dots\right)\ of\ (y_m)$ whose terms are of the form

$$y_{n,m} = \sum_{j=1}^{n} \lambda_j{}^{(m)} x_j \quad ; \quad \left(\sum_{j=1}^{n}\left|\lambda_j{}^{(m)}\right| = 1\right)$$

with scalars $\lambda_j{}^{(m)}$ satisfying $\lambda_j{}^{(m)} \longrightarrow \beta_j$ as $m \longrightarrow \infty$ .So that , as $m \longrightarrow \infty$, $y_{n,m} \longrightarrow y = \sum_{j=1}^{n} \beta_j x_j$ where $\sum\left|\beta_j\right| = 1$,hence not all β_j can be zero . Since

$\{x_1, x_2, \dots, x_n\}$ is a linearly independent set , we have $y \neq 0$. On the other hand , $y_{n,m} \longrightarrow y$ implies $\left\|y_{n,m}\right\| \longrightarrow \|y\|$. Since $\|y_m\| \longrightarrow 0$ and $\left(y_{n,m}\right)$ is a subsequence of (y_m) , we must have $\left\|y_{n,m}\right\| \longrightarrow 0$. Hence $\|y\| = 0$,thus $y = 0$. This contradicts $y \neq 0$.

Theorem (1.3.2) [20]

If a normed space X is finite dimensions subspaces Y are all complete.

Proof

Let (y_m)be a Cauchy sequence in Y, and y will represent the limit. If dim Y$= n$ and any basis for Y, $\{e_1, .., e_n\}$. Then $y_m$has a unique representation of the form.

$$y_m = \alpha_1{}^m e_1 + \cdots + \alpha_n{}^m e_n$$

From imposition , any $\in > 0$ exist N thus
$\|y_m - y_r\| < \in$ when $m, r > N$ .By lemma (1.3.1) we have

$$\in > \|y_m - y_r\| = \left\|\sum_{j=1}^{n}\left(\alpha_j{}^{(m)} - \alpha_j{}^{(r)}\right)e_j\right\| \geq c\sum_{j=1}^{n}\left|\alpha_j{}^{(m)} - \alpha_j{}^{(r)}\right|$$

Division by $c > 0$ produces where $m, r > N$ we obtian

$$|\alpha_j{}^{(m)} - \alpha_j{}^{(r)}| \leq \sum_{j=1}^{n} |\alpha_j{}^{(m)} - \alpha_j{}^{(r)}| < \frac{\in}{c} \qquad (m, r > N)$$

For every of the n sequences

$(\alpha_j{}^{(m)}) = (\alpha_j{}^{(1)}, \alpha_j{}^{(2)}, \dots)$ $j = 1, \dots, n$ Is Cauchy in R or C? In order for it to converge, if α_j denotes the limit. Using these n limits therefore, $\alpha_1, \dots \alpha_n$ we define

$$y = \alpha_1 e_1 + \dots + \alpha_n \mathrm{e_n}$$

hence $y \in Y$,

$\|y_m - y\| = \|\sum_{j=1}^{n} (\alpha_j{}^{(m)} - \alpha_j) e_j\| \leq \sum_{j=1}^{n} |\alpha_j{}^{(m)} - \alpha_j| \, \|e_j\|$.

On the right , $\alpha_j{}^{(m)} \longrightarrow \alpha_j$. Hence $\|y_m - y\| \longrightarrow 0$ this is , $y_m \longrightarrow y$. This shows that (y_m) is convergent in Y . Thus Y is complete .

1.4 Linear Operators

Let X and Y be finite dimensional vector spaces and X,Y in field K. If $T(\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in K$, T is also a linear operator, the mapping $T: X \to Y$ is said to be linear.

If $\dim(X) = n$ and $\dim(Y) = m$, choose two a basiss $\{e_1, e_2, \dots, e_n\}$ for X and $\{f_1, f_2, \dots, f_m\}$ for Y. The following is how a linear operator $T: X \to Y$ corresponds to a $m \times n$ matrix A of elements of f [19].

Definition (1.4.1)

A linear operator T is mapping $T: X \longrightarrow Y$ when X and Y are vector spaces defined on the same field K.

1) Let domain $D(T)$ of T is a vector space and that $R(T)$ is a range in the same field.

2) $\forall\, x, y \in D(T)$ and $\forall\, \alpha \in K$,

$$T(x+y) = Tx + Ty$$

$$T(\alpha x) = \alpha T x.$$

Furthermore for the remainder $N(T)$ is the null space of T. Since $N(T)$ is the set of all $x \in D(T)\ such\ that\ Tx = 0$.

Next we give some examples of linear operators. $x \in D(T)\ such\ that\ Tx = 0$

Examples(1.4.1)

1) **The zero operator spaces.** The operator $O: X \to Y$ is defined by

$O(x) = 0$ for all $x \in X$

2) **Differentioation.** If X is all polynomials on $[a, b]$ and define a linear operator T on X given

$$Tx(t) = \acute{x}(t)$$

for each $x \in X$, when the prime indicates differentiation from t .Such that maps $T: X \longrightarrow X$.

3) **Integration space.** If define $T: C[a, b] \longrightarrow C[a, b]$; T is a linear operator defined by

$$Tx(t) = \int_a^t x(t)dt \qquad ; t \in [a, b]$$

4) **Matrices.** Let a real matrix $A = [a_{ij}]$ with m rows and n columns defines an operator $T: \mathbb{R}^n \to \mathbb{R}^m$ by

$$y = Ax$$

Due to the standard practice of matrix multiplication, when $x = (x_i)$ has n components and simlary $y = (y_i)$ has m , both vectors are written as column vectors; writing $y = Ax$,thus

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Since matrix multiplication is a linear operation, hence T is linear

Thus the linearity is used is proofs [6].

Theorem (1.4.1) [13]

If T is a linear operator then If $dim\ D(T) = n < \infty$,$T: X \longrightarrow Y$

$T: D(T) \longrightarrow R(T)$ .Then dim $R(T) \leq n$.

Proof

We choose n+1 elements $y_1, \dots, y_{n+1}$ of (T) .

Then we have

$$y_1 = Tx_1, \dots, y_{n+1} = Tx_{n+1}$$

For some $x_1, \dots, x_{n+1}$ in $D(T)$. Since $dimD(T) = n$,this set

$\{x_1, \dots, x_{n+1}\}$ must be linearly dependent . Hence

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$$

$\exists\ \alpha_1, \dots, \alpha_{n+1}$, no every equal 0.

Because T be linear and $T0 = 0$

$$T(\alpha_1 x_1 + \cdots \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \cdots + \alpha_{n+1} y_{n+1} = 0.$$

The fact that the α_j`s are not all zero demonstrates that the set $\{y_1, \dots, y_{n+1}\}$ is linearly dependent. Hence R(T) subsets of n+1 or more components that are no linearly independent .Thus $dimR(T) \leq n$.

Definition (1.4.2)

Let $T: D(T) \longrightarrow Y$ be a linear operator is said to be **injective** or one to one if for any$x_1, x_2 \in D(T)$, $x_1 \neq x_2 \Longrightarrow Tx_1 \neq Tx_2$.

There exists the mapping

$$T^{-1}: R(T) \longrightarrow D(T)$$

$$y_0 \longmapsto x_0 \quad (y_0 = Tx_0).$$

Which maps every $y_0 \in R(T)$, $x_0 \in D(T)$ for which $Tx_0 = y_0$ .The mapping T^{-1} is called the inverse of T.

We clearly have $T^{-1}Tx = x \;\; for\ all\ x \in D(T)$

Lemma(1.4.2) [13]

If $T: X \longrightarrow Y$ and $S: Y \longrightarrow Z$ are bijective linear operators , where X, Y, Z, are vector spaces. Then the inverse $(ST)^{-1}: Z \longrightarrow X$ of the product (the composite) ST exists, and

$$(ST)^{-1} = T^{-1}S^{-1}$$

Proof

The operator $ST: X \longrightarrow Z$ is bijective ,so that $(ST)^{-1}$ exists. Such that

$$\mathrm{ST}(ST)^{-1} = I_z$$

whereI_z is (the identity operator on Z) .

stratifying S^{-1} and using $S^{-1}S = I_y$,

we have

$$S^{-1}ST(ST)^{-1} = T(ST)^{-1} = S^{-1}I_z = S^{-1}$$

Applying T^{-1} and using $T^{-1}T = I_x$

We obtain that

$$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1} .$$

1.5 Bounded Linear Operators.

"Between the one-dimensional scalar field beneath the linear space and every linear functional, there is a linear operator". [5].

Definition (1.5.1)

If X and Y are normed space and $T: D(T) \longrightarrow Y$be a linear operator , where $D(T) \subset X$. If real number c exists and such that for any $x \in D(T)$, then T be called a bounded.

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

$$\|Tx\| \leq c\|x\|.$$

A bounded linear operator translates bounded sets in $D(T)$ onto bounded sets in Y, as demonstrated by definition (1.5.1).

Next we give some examples .

Examples (1.5.1)

1) Let $I: X \to X$ is the identity operator on a normed space where $X \neq \{0\}$ is bounded and when $\|I\| = 1$.

2) Consider examples(1.4.1) part (4)

$$y = Ax$$

Where $$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad and \;\; Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Note that

$$Y_j = \sum_{k=1}^{n} a_{jk} x_k \qquad (j = 1, \dots, m)$$

Since T is linear

Note that the norm on $\mathbb{R}^n$ is given by

$$\|x\| = \left(\sum_{i=1}^{n} {x_i}^2\right)^{\frac{1}{2}}$$

Similarly for $y \in \mathbb{R}^m$.

we thus obtain

$$\|Tx\|^2 = \sum_{i=1}^{m} {y_i}^2 = \sum_{j-1}^{m} \left[\sum_{k=1}^{n} a_{jk} x_k\right]^2$$

$$\leq \sum_{i-1}^{m} \left[\left(\sum_{k=1}^{n} {a_{jk}}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} {x_k}^2\right)^{\frac{1}{2}}\right]^2$$

$$= \|x\|^2 \sum_{j=1}^{m} \sum_{k=1}^{n} {a_{jk}}^2$$

Thus

$$\|Tx\|^2 \leq c^2 \|x\|^2 \quad where \quad c^2 = \sum_{j=1}^{m} \sum_{k=1}^{n} {a_{jk}}^2.$$

Then

$$\|Tx\| \leq c\|x\|$$

Implies that T is bounded .

Theorem (1.5.1) [19]

Let $T: D(T) \longrightarrow Y$ be linear operator , where $D(T) \subset X$ and X, Y be normed spaces. Then

1) If and only if T is bounded, T is continuous.

Proof

1) for $= 0$. let $\neq 0$. Then $\|T\| \neq 0$.suppose T to be bounded , if any $\varepsilon >= 0$ by provided. Since T is linear, this means that for all $x_0, x \in D(T)$

like that

$$\|x - x_0\| < \delta \qquad \text{when} \qquad \delta = \frac{\epsilon}{\|T\|}$$

Thus

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \epsilon.$$

Since $x_0 \in D(T)$, hence T is continuous .

Conversely , supposing T is continuous at any given $x_0 \in D(T)$.

Then there is a $\delta > 0$ given any $\varepsilon > 0$ so that

$\|Tx - Tx_0\| \leq \epsilon$ for every $x \in D(T)$ satisfying $\|x - x_0\| < \delta$.

Now take any $y \neq 0$ in $D(T)$ and set

$x = x_0 + \frac{\delta}{\|y\|} y$ .Then $x - x_0 = \frac{\delta}{\|y\|} y$.

Hence $\|x - x_0\| = \delta$. since T is linear , we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\left(\frac{\delta}{\|y\|} y \right)\right\| = \frac{\delta}{\|y\|} \|Ty\|$$

implies $\frac{\delta}{\|y\|} \|Ty\| \leq \epsilon$. Thus $\|Ty\| \leq \frac{\epsilon}{\delta} \|y\|$.

This can be written $\|Ty\| \leq c\|y\|$,where $c = \frac{\epsilon}{\delta}$ and T is bounded .

1.6 Linear Functionals

Definition (1.6.1)

If f is a linear functional, then f is a linear operator with a range in the scalar field K of X and a domain in a vector space X, hence

$$f: D(f) \longrightarrow K$$

When $K = C$ if X is complex and $K = R$ if X is real.

Definition (1.6.2)

Let's say that the domain $D(f)$ lies in the scalar field of the normed X and that the bounded linear functional f is bounded linear operator with rang. In light of this, real integer c exists such that for any $y \in D(f)$.

$$|f(y)| \leq c\|y\|$$

The norm of f is

$$\|f\| = sup_{y \in D(f)\,;y \neq 0} \quad \frac{|f(y)|}{\|y\|}$$

or

$$\|f\| = sup_{y \in D(f)\,;\|y\|=1} \quad |f(y)|$$

This implies $|f(y)| \leq \|f\|\|y\|$,

Next we give some examples.

Examples(1.6.1)

1)If $X = C[a, b]$ **,** then

$$f(x) = \int_a^b x(t)\, dt \qquad x \in C[a, b]$$

f is linear functional. shows that f is bounded and has$\|f\| = b - a$. In fact, writing $J = [a, b]$ and remembering the norm on $C[a, b]$,

We obtain

$$|f(x)| = \left|\int_a^b x(t)\, dt\right| \leq \int_a^b x(t)\, dt\ .$$

Since we have

$$M_j = \sup\{f(x); x_{j-1} \leq x \leq x_j\}$$

$$M = \max\{x(t); a \leq t \leq b\}$$

and since

$$f(x) \leq |f(x)|, \quad a \leq x \leq b.$$

Thus

$$|f(x)| = \left|\int_a^b x(t)\, dt\right| \leq \int_a^b x(t)\, dt$$

$$= (b-a) \max_{t \in J} |x(t)|$$

$$=(b-a)\|x\|.$$

By definition (1.6.2) we obtain $\|f\| \leq b - a$.

We choose $x = x_0 = 1$,

note that $\|x_0\| = 1$

$$\|f\| \geq \frac{|f(x_0)|}{\|x_0\|} = |f(x_0)| = \int_a^b dt = b - a\ .$$

This implies

$$\|f\| = b - a.$$

Thus, f be bounded linear functional .

Definition (1.6.3)

A collection of each and every linear functional defined on the vector space X. The definition of the vector space's under algebraic operations is as follows.

a) $The\ sum\ f_1 + f_2$ of two functionals f_1 and f_2 is the functional whose value at every $x \in X$ is

$$s(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x).$$

b) The functional P is the product αf of a scalar α and a functional f, and its value at $x \in X$ be

$$P(x) = (\alpha f)(x) = \alpha f(x).$$

Thus X^* is said to the **algeberaic dual space of X**

Definition (1.6.4)

The algebraic dual $(X^*)^*$ of X^*whose members are the linear functionals defined on X^* $such\ that$ X^{**} is referred to as **the second algeberaic dual space of X** if a collection of all linear functionals defined on a vector space X.

Definition (1.6.5)

if the space X is normed. The norm of the normed space formed by the set of all bounded linear functionals on X is defined as

$$\|f\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|=1}} |f(x)|$$

Since $\grave{X}$ is said to be the dual space of X . We have $\grave{X}$ is $B(X,Y)$ with the complete space $Y = \mathbb{R}$ or $\mathbb{C}$ because a linear functional on X maps X into $\mathbb{R}$ or $\mathbb{C}$ (the scalar field of X) and sine R or C, taken with a metric, is complete.

Chapter Two

HILBERT SPACE

A Hilbert space is made up of a vector space and an inner product that gives it the structure of an entire metric space.

The reader is already familiar with the intermingling of algebra and geometry, namely in the vector space $\mathbb{R}^n$, elements in $\mathbb{R}^n$ [21].

Typically, points have coordinates and vectors can be added and scaled .Moreover , in the presence of the standard inner product, since X is normed space ; given by

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k \ ; x, y \in X$$

the length of a vector provided by the norm

$$\|y\| = \sqrt{\langle y, y \rangle}$$

and angle between vectors can be computed by

$$\theta = arc \cos \frac{\langle x,y \rangle}{\|x\| \|y\|} .$$

and the condition for orthogonality $a.b = 0$

which are important tools in many applications [4].

2.1 Inner Product Spaces .

Definition (2.1.1)

A vector space X with an inner product defined on X is known as an inner product space (or pre Hilbert space).

An inner product on X in this context is a mapping of $X \times X$ into the scalar field K of X. For each pair of vectors x and y, denoted as $\langle z, w \rangle$and is said to the **Inner product** of z and w, such that for all vectors z, w, and v and scalars α, we have

1) $\langle z + w, v \rangle = \langle z, v \rangle + \langle w, v \rangle$

2) $\langle \alpha z, w \rangle = \alpha \langle z, w \rangle \ and \ \langle z, \beta w \rangle = \bar{\beta} \langle z, w \rangle$

3) $\langle z, w \rangle = \overline{\langle z, w \rangle}$

4) $\langle z, z \rangle \geq 0 \ and \ \langle z, z \rangle = 0 \iff z = 0.$

A metric on X defined by the expression

$$d(z,w) = \|z-w\| = \sqrt{\langle z-w, z-w\rangle}$$

is called an inner product on X.

The conjugation of the bar complex is in (3). Let X be a real vector space, then

$$\langle z,w\rangle = \langle w,z\rangle$$

The part (2) denotes

$$\langle z,\alpha w\rangle = \alpha\langle w,z\rangle$$

Definition (2.1.2)

A complete inner product space is a Hilbert space.

Easy consequences

1) $\|x+y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2$.

Proof

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2$$

2) **(pythogoras)** If $\langle y,z\rangle = 0$, then $\|y+z\|^2 = \|y\|^2 + \|z\|^2$.

Proof

Since $\|y+z\|^2 = \|y\|^2 + 2Re\langle y,z\rangle + \|z\|^2$ by (1) then

We have $\langle y,z\rangle = 0$, thus $\|y+z\|^2 = \|y\|^2 + \|z\|^2$.

More generally if $\langle y_i, y_j\rangle = 0\ for\ i \neq j$, then $\|y_1 + \cdots + y_N\|^2 = \|y_1\|^2 + \cdots + \|y_N\|^2$.

Definition (2.1.3)

When $\langle x,y\rangle = 0$, it is said that an element x of an inner product space X is orthogonal to an element y$\in X$. And say that x and y are orthogonal, write $x \perp y$. Also for subsets $A, B \subset X$ we write $x \perp A\ \ if\ x \perp a$ and $A \perp B\ if\ a \perp b$ for all $a \in A$ and all $b \in B$.

Do all norms on vector spaces come from inner products , and if not , which property characterizes inner product spaces ?

We obtain answer by **parallelogram law** [19].

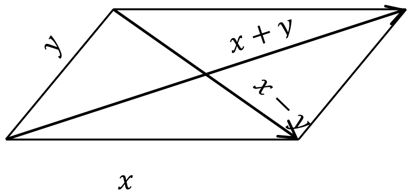

Fig .1. Parallelogram with sides x and y in the plane

Theorem (2.1.1) [15]

For each vector x, y, the inner product induces a norm if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Proof

The parallelogram law follows from adding the identities ,

$$\|x+y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2 ,$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle + \langle x,-y\rangle + \langle -y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 - 2Re\,\langle x,y\rangle + \|y\|^2.$$

Subtracting the two gives $4Re\,\langle x,y\rangle$. This is sufficient to identify the inner product when the scalar field is $\mathbb{R}$. $Over$ $\mathbb{C}$ notice that $Im\,\langle x,y\rangle = -Re\; i\langle x,y\rangle = Re\,\langle ix,y\rangle\,, so$

$$\langle x,y\rangle = \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2).$$

Define for any normed space ,

$$\langle\langle x,y\rangle\rangle := \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2),$$

for a complex space , $\langle x,y\rangle \coloneqq \langle\langle x,y\rangle\rangle + i\langle\langle ix,y\rangle\rangle$.

So that $\langle\langle y,x\rangle\rangle = \langle\langle x,y\rangle\rangle$ and $\langle x,x\rangle = \langle\langle x,x\rangle\rangle = \|x\|^2$, as well as $\langle x,0\rangle = \langle\langle x,0\rangle\rangle = 0$; $\langle y,x\rangle = \overline{\langle x,y\rangle}$ is readily verified by

$$4\langle\langle iy, x\rangle\rangle = \|x + iy\|^2 - \|x - iy\|^2 = \|y - ix\|^2 - \|y + ix\|^2 = -4\langle\langle ix, y\rangle\rangle\ .$$

If the parallelogram law is satisfied is the hardest part of the proof then Showing that linearity holds. Writing

$$2y \pm x = (y + z \pm x) + (y - z),$$

$$2z \pm x = (y + z \pm x) - (y - z),$$

and using the parallelogram law ,

$$4\langle\langle x, 2y\rangle\rangle + 4\langle\langle x, 2z\rangle\rangle = \|2y + x\|^2 - \|2y - x\|^2 + \|2z + x\|^2 - \|2z - x\|^2$$

$$= \|2y + x\|^2 + \|2z + x\|^2 - \|2y - x\|^2 - \|2z - x\|^2$$

$$= 2\|y + z + x\|^2 + 2\|y - z\|^2 - 2\|y + z - x\|^2 - 2\|y - z\|^2$$

$$= 8\langle\langle x, y + z\rangle\rangle\ .$$

Putting z=0 gives $\langle\langle x, 2y\rangle\rangle = 2\langle\langle x, y\rangle\rangle$, reducing the above identity to

$$\langle\langle x, y + z\rangle\rangle = \langle\langle x, y\rangle\rangle + \langle\langle x, z\rangle\rangle$$

Thus $\langle\langle x, ny\rangle\rangle = n\langle\langle x, y\rangle\rangle$ for n ∈ ℕ . For the negative integers,

$$\langle\langle x, -y\rangle\rangle = \|-y + x\|^2 - \|-y - x\|^2 = -\langle\langle x, y\rangle\rangle$$

while for rational numbers $P = m/n\ , m, n \in\ \mathbb{Z}\,, n \neq 0,$

$$n\,\langle\langle x, \tfrac{m}{n}y\rangle\rangle = \langle\langle x, my\rangle\rangle = m\langle\langle x, y\rangle\rangle$$

so

$$\langle\langle x, py\rangle\rangle = p\langle\langle x, y\rangle\rangle.$$

Note that $\langle\langle x, y\rangle\rangle$ is continuous in x and y since the norm is continuous , so if the rational numbers $P_n \longrightarrow \alpha \in \mathbb{R}$, then

$$\langle\langle x, \alpha y\rangle\rangle = \lim_{n\to\infty}\langle\langle x, P_n y\rangle\rangle = \lim_{n\to\infty} P_n\langle\langle x, y\rangle\rangle = \alpha\langle\langle x, y\rangle\rangle.$$

Now over the complex numbers , $\langle x, \beta y\rangle = \beta\langle x, y\rangle \ for\ \beta \in \mathbb{C}$, and $\langle x, iy\rangle = -\langle\langle ix, y\rangle\rangle + i\langle\langle x, y\rangle\rangle = i\langle x, y\rangle$.

Hence, A norm cannot be generated from an inner product if it does not meet the parallelogram equality condition. Can be write

Not all normed space are inner product space .

Note that example (3).

We have already seen that the inner product space $\mathbb{R}$ with $\langle x, y\rangle = xy$ and hence $\|x\| = |x|$ is a (one dimensional) Hilbert space – that is to say, every Cauchy sequence of real numbers is convergent.

It is easily seen that a sequence of complex numbers, $(a_n + ib_n)$, is a Cauchy [convergent] sequence if and only if both the sequence of real parts ,(a_n), and the sequence of imaginary parts, (b_n), are Cauchy [convergent] sequences. Thus, $\mathbb{C}$ since $\langle x, y\rangle = x\bar{y}$ and hence $\|x\| = |x|$ is a Hilbert space[17]

Examples (2.1.1)

1)$\mathbb{R}^n$, $\mathbb{C}^n$ are all Hilbert space.

$\mathbb{C}^n$ and $\mathbb{R}^n$ taken to be the standard inner product, $\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}$ are both complete. To see this (for $\mathbb{C}^n$,the proof for $\mathbb{R}^n$is essentially the same),let $(x_m)_{m=1}^{\infty}$ be a Cauchy sequence in $\mathbb{C}^n$,so each $x_m$is an n-tuple of complex numbers ; $x_m = (x_{m1}, x_{m2}, \dots, x_{mn})$.

We need to show that $(x_m)$is convergent. Now, for each $k \in \mathbb{N}$ we have,

$$|x_{mk} - x_{pk}| = \sqrt{|x_{mk} - x_{pk}|^2} \leq \sqrt{\sum_{i=1}^{n} |x_{mi} - x_{pi}|^2}$$

$$= \|x_m - x_p\| \longrightarrow 0, \text{as } m, p \longrightarrow \infty,$$

Since (x_m) is a Cauchy . This shows that for each $k \in \{1,2, \dots, n\}$ the sequence of k`th components, $(x_{mk})_{m=1}^{\infty}$,is a Cauchy sequence of complex numbers and hence (by the completeness of $\mathbb{C}$) convergent.

Let $x_k = \lim_m x_{mk}$.

We have

$$X_1 = (x_{11}, x_{12}, x_{13}, \dots, x_{1n})$$

$$X_2 = (x_{21}, x_{22}, x_{23}, \dots, x_{2n})$$

$$X_3 = (x_{31}, x_{32}, x_{33}, \dots, x_{3n})$$

$$\vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$X_m = (x_{m1}, x_{m2}, x_{m3}, \dots, x_{mn})$$

$$\vdots \qquad \downarrow \quad \downarrow \quad \downarrow \quad , \dots, \downarrow$$

$$\vdots \qquad x_1 \quad x_2 \quad x_3, \dots, x_m$$

Now ,let $X = (x_1, x_2, x_3, \dots, x_n)$. Finally, we show that $X_m \longrightarrow x$. To this end ,note that

$$\lim_m \|X_m - x\| = \lim_m \sqrt{\sum_{k=1}^{n} |x_{mk} - x_k|^2}$$

$$= \sqrt{\sum_{k=1}^{n} \left(\lim_m |x_{mk} - x_k|\right)^2} \ ,$$

Since $\lim_m |x_{mk} - x_k| = 0$,for $k = 1,2, \dots, n$.Thus, (X_m) is convergent (to x), as required.

2) Space $l^2$are Hilbert space, Hilbert sequence space l^2(1912) (Integral Equations)

l^2, the space of square summable complex (or real) sequences with the inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$, is complete. In many ways, we can regard l^2 as the Hilbert space. The proof similar to that for $\mathbb{C}^n$ given above.

Let (x_n) be a Cauchy sequence in l^2 , where $x_n = ({x_n}^1, {x_n}^2, \dots, {x_n}^3, \dots)$; that is , for each $n \in \mathbb{N}$ we have $\sum_{k=1}^{\infty} |{x_n}^k|^2 \leq \infty$ and $\|x_n - x_m\| \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Then ,as above, for each k,

$$\|{x_n}^k - {x_m}^k\| = \sqrt{\sum_{i=1}^{\infty} |{x_n}^i - {x_m}^i|^2}$$

$$= \|x_n - x_m\| \longrightarrow 0$$

as (x_n) is Cauchy so for each k, $(x_n)^k$ is a Cauchy sequence of (real or complex) numbers and hence convergent, to say x^k.

Let $x = (x^1, x^2, x^3, \dots, x^k, \dots)$. To complete the proof we show that $x \in l^2$ and that (x_n) converges to x. Firstly, consider the partial sum $\sum_{k=1}^{m} |x_k|^2$ for each $m \in \mathbb{N}$. Being a sum of non-negative terms, this sum is increasing, so the partial sums will converge if they are bounded from above. Now,

$$\sum_{k=1}^{m} |x_k|^2 = \sum_{k=1}^{m} \left|\lim_{n} x_n{}^k\right|^2 = \lim_{n} \sum_{k=1}^{m} |x_n{}^k|^2,$$

$$\leq \lim_{n} \sum_{k=1}^{\infty} |x_n{}^k|^2 = \lim_{n} \|x_n\|.$$

That this last limit exist and is finite (and hence provides an upper bound for the partial sums) follows from the observation that $(\|x_n\|)$ is real Cauchy sequence , since $(x_n)$is Cauchy $(|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\| \longrightarrow 0)$ and so convergent. Finally, we establish the convergence of (x_n)in l^2 by showing that $x_n \longrightarrow x$. Now, for any $\epsilon > 0$, since (x_n) is Cauchy, there exists an $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$ whenever $m, n \geq n_0$. Thus, for each $q \in \mathbb{N}$,we observe that

$$\sqrt{\sum_{k=1}^{q} |x_n{}^k - x_m{}^k|^2} \leq \sqrt{\sum_{k=1}^{\infty} |x_n{}^k - x_m{}^k|^2}$$

$$= \|x_n - x_m\| < \epsilon, \text{ proved } m, n \geq n_0.$$

But then, for $n \geq n_0$ we have,

$$\|x_n - x\| = \lim_{q} \sqrt{\sum_{k=1}^{q} |x_n{}^k - x^k|^2}$$

$$= \lim_{q} \sqrt{\sum_{k=1}^{q} \left|x_n{}^k - \left(\lim_{m} x_m{}^k\right)\right|^2}$$

$$= \lim_{q} \lim_{m} \sqrt{\sum_{k=1}^{q} |x_n{}^k - x_m{}^k|^2}$$

$$\leq \epsilon,$$

Showing that $x_n \longrightarrow x$.

3) Space l^p. Is not a Hilbert space because with $P \neq 2$ is not an inner product space.

We demonstrate that the norm does not satisfy theorem (2.1.1).

Let

$$x = \{x_n\}_{n=1}^{\infty} \quad and \ \ y = \{y_n\}_{n=1}^{\infty}$$

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

Define inner product space on l^p such that

$$\|x\| = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{\frac{1}{p}}$$

Let us take $x = \{1,0,0,0, \dots\} \in l^p \ \ and \ \ y = \{1, -1, 0,0,0, \dots\} \in l^p$and cakulate

$$\|x\| = \|y\| = 2^{\frac{1}{p}} \ \ , but \ \ \|x + y\| = \|x - y\| = 2$$

We now see that parallelogram equality is not satisfied if $P \neq 2. Then\ l^p\ is$ complete .

4) Space $C[a, b]$ .The space $C[a, b]$ be not an inner product space so that not a Hilbert space .

Suppose that

$$\|y\| = \max_{t \in J} |y(t)| \qquad J = [a, b]$$

This equality , Cannot be obtained from an inner product because does not satisfy the parallelogram law .If we adopt

$$y(t) = 1 \ \ and \ \ z(t) = (t-a)/(b-a),$$

we have

$$\|y\| = 1 \,, \|z\| = 1 \quad and$$

$$y(t) + z(t) = 1 + \frac{t-a}{b-a}$$

$$y(t) - z(t) = 1 - \frac{t-a}{b-a}.$$

Hence $\|y+z\| = 2 \,, \|y-z\| = 1 \ \ and$

$$\|y+z\|^2 + \|y-z\|^2 = 5 \quad but \quad 2(\|y\|^2 + \|z\|^2) = 4.$$

5)For an inner product space over $\mathbb{C}$. if $\langle y, Ty \rangle = 0 \ \ for\ all \ \ y \in X \,, then\ T = 0$.

The identities

$$0 = \langle y+z, T(y+z) \rangle = \langle y, Tz \rangle + \langle z, Ty \rangle,$$

$$0 = \ \langle y+iz, T(y+iz) \rangle = i\langle y, Tz \rangle - i\langle z, Ty \rangle,$$

Together empty $\langle y, Tz \rangle = 0$ for any $y, z \in X$ inparticular $\|Tz\|^2 = 0$.

2.2 Some Properties of Inner Product Spaces

In this section we will show some definition and theorems .

Theorem (2.2.1) [13]

Let X is an inner product , then

1) $|\langle z, w \rangle| \leq \|z\|\|w\| \qquad (schwarz\ inequality)$

When and only when $\{z, w\}$is a set that is linearly dependent, the equality sign is present

2) That norm satisfies

$$\|z+w\| \leq \|z\| + \|w\| \qquad (Triangle\ inequality)$$

Proof

1) If $w = 0$, then schwarz inquality holds since $\langle z, 0\rangle = 0$.

If $w \neq 0$. To any scalar α such that

$$0 \leq \|z - \alpha w\|^2 = \langle z - \alpha w\ , z - \alpha w\rangle$$

$$= \langle z, z\rangle - \bar{\alpha}\langle z, w\rangle - \alpha[\langle w, z\rangle - \bar{\alpha}\langle w, w\rangle].$$

since $[\langle w, z\rangle - \bar{\alpha}\langle w, w\rangle] = 0$, if choose $\bar{\alpha} = \langle w, z\rangle/\langle w, w\rangle$.

The remaining inequality is

$$0 \leq \langle z, z\rangle - \frac{\langle w, z\rangle}{\langle w, w\rangle}\langle z, w\rangle = \|z\|^2 - \frac{|\langle z, w\rangle|^2}{\|w\|^2}\ ;$$

here we used $\langle w, z\rangle = \overline{\langle z, w\rangle}$. Multiplying by $\|w\|^2$, taking square roots, we obtain(1).

$$w = 0\ or\ \ 0 = \|z - \alpha w\|^2,$$

thus $z - \alpha w = 0$, $so\ that\ \ z = \alpha w$, which shows linear dependence .

2) Where $w = 0\ or\ z = cw\ \ (c\ real\ \ and\ \ \geq 0)$ are the only conditions under which the equality sign is true.

we have

$$\|z+w\|^2 = \langle z + w\ , z + w\rangle = \|z\|^2 + \langle z, w\rangle + \langle w, z\rangle + \|w\|^2.$$

By part (1) in theorem ,

And from the triangle inequality we get on

$$\|z+w\|^2 \leq \|z\|^2 + 2|\langle z, w\rangle| + \|w\|^2$$

$$\leq \|z\|^2 + 2\|z\|\|w\| + \|w\|^2$$

$$= (\|z\| + \|w\|)^2.$$

we obtain (2).

In this derivation equality holds iff

$$\langle z, w\rangle + \langle w, z\rangle = 2\|z\|\|w\|.$$

From part (1) and $2Re\langle z, w\rangle$ is written on the left side, when Re stands for the real part.

$$Re\langle z, w\rangle = \|z\|\|w\| \geq |\langle z, w\rangle| \longrightarrow (3)$$

Because the real component of a complex number cannot be more than its absolute value, we have equality, which implies dependence by part (1)

so, $w = 0 \ or \ z = cw$.

Demonstrate that $c \geq 0$ and be real.

From (3) and the equality sign

we obtain

$$Re\langle z, w\rangle = |\langle z, w\rangle|.$$

However, the imaginary portion of a complex number must be 0 if the real part of the number equals its absolute value.

hence $\langle z, w\rangle = Re\langle z, w\rangle \geq 0$ by, (3) and $c \geq 0$

Thus

$$0 \leq \langle z, w\rangle = \langle cw, w\rangle = c\|w\|^2.$$

The Schwarz inequality can be used in proofs follwoing.

Corollary(2.2.2) [20]

Let an inner product space is X and $\|\cdot\| is$ the induced norm , then

$$\|z\| = \sup_{\|y\|\leq 1} |\langle z, y\rangle| = \sup_{\|y\|=1} |\langle z, y\rangle|$$

For all $z \in X$.

Proof

If $z = 0$ the assertion is obvious , so suppose that $z \neq 0$. If $\|y\| \leq 1$, then $|\langle z, y\rangle| \leq \|z\|\|y\| = \|z\|$, from theorem (2.2.1) part (1). Hence

$$\|z\| \leq \sup_{\|y\| \leq 1} |\langle z, y\rangle|.$$

Choosing $y = z/\|z\|$ we have $|\langle z, y\rangle| = \|z\|^2/\|z\| \leq \|z\|$,So equality hold in the above inequality. Since the supremum over $\|y\| = 1$ is larger or equal to that over $\|y\| \leq 1$,the assertion of the corollary follows .

Theorem(2.2.3) [11]

The norm in an inner product space is strictly $(\|w\| > 0 \; whenever \; w \neq 0)$,

Positively homogeneous $(\|\alpha w\| = |\alpha|\|w\|)$, subadditive $(\|w + z\| \leq \|w\| + \|z\|)$.

Proof

The strict positiveness of the norm is merely a restatement of strict positiveness of the inner product .

The positive homogeneity of the norm is a consequence of the identity

$$\|\alpha w\|^2 = \langle \alpha w, \alpha w\rangle = \alpha\alpha^*\langle w, w\rangle = |\alpha|^2\|w\|^2$$

The subadditivity of the norm follows,

using Schwarz`s inequality ,from the relations

$$\|w + z\|^2 = \langle w + z, w + z\rangle \leq \|w\|^2 + |\langle w, z\rangle| + |\langle z, w\rangle| + \|z\|^2$$

$$\leq \|w\|^2 + 2\|w\|\|z\| + \|z\|^2$$

$$= (\|w\| + \|z\|)^2$$

$$\|w + z\| \leq \|w\| + \|z\|.$$

Lemma(2.2.4) [20]

Let $z_n \longrightarrow z$ and $w_n \longrightarrow w$ in an inner product space then

$$\langle z_n, w_n \rangle \longrightarrow \langle z, w \rangle.$$

Proof

Using the theorem(2.2.1) part (1)and part (2) we have

$$|\langle z_n, w_n \rangle - \langle z, w \rangle| = |\langle z_n, w_n \rangle - \langle z_n, w \rangle + \langle z_n, w \rangle - \langle z, w \rangle|$$

$$\leq |\langle z_n, w_n - w \rangle| + |\langle z_n - z, w \rangle|$$

$$\leq \|z_n\| \|w_n - w\| + \|z_n - z\| \|w\| \longrightarrow 0$$

Since $w_n - w \longrightarrow 0 \ and \ z_n - z \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty$.

Then $|\langle z_n, w_n \rangle - \langle z, w \rangle| \longrightarrow 0$.

2.3 Orthogonal and Orthonormal sets.

The distance d between an element in a metric space $x \in X$ and a nonempty subset $M \subset X$ is defined as

$$d = inf_{\hat{y} \grave{\in} M} \, d(x, \hat{y})$$

Becomes in a normed space

$$d = inf_{y \in M} \, \|x - \hat{y}\|$$

We shall show that it is crucial to know whether a $y \in M$ exists, so that

$$d = \ \|x - y\|.$$

We show some definitions and theorem [13].

Definition (2.3.1)

A segment joining given by $z = \alpha x + (1 - \alpha) y \quad (\alpha \in R \, , 0 \leq \alpha \leq 1)$

is two elements x and y of a vector space X is defined the set of every $z \in X$.

Definition (2.3.2)

If the segment joining $x \ and \ y$ is contained in M for any $x, y \in M$, then the subset M of X is said to be **convex**.

If M is a convex set then the theorem(2.3.1) answers on the previous questions .

Definition(2.3.3)

If N is a normed space and M a non-empty closed subset. We define the set of projections of y onto M by

$$P_M(y) = \{m \in M: \|y - m\| = dist(y, M)\}.$$

The meaning of $P_M(y)$ be illustrated in Figure(2) for the Euclidean norm in the plane.

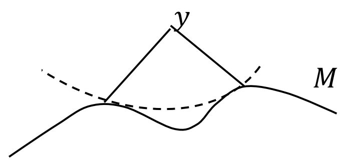

Fig.2. The set of nearest point Projections

Theorem (2.3.1) [13]

If $M \neq \emptyset$ is a complete convex subset and X be an inner product space. Then, there exists a unique $y \in M, \forall\, x \in X$ so that

$$d = inf_{y \in M} \|x - \hat{y}\| = \|x - y\|.$$

Proof

1) There is sequence (y_n) in M by the definition of an infimum

hence

$$d_n \longrightarrow d \quad where \;\; d_n = \|x - y_n\|.$$

Let $y_n - x = v_n$, we obtain $\|v_n\| = d_n$ and

$$\|v_n + v_m\| = \|y_n + y_m - 2x\| = 2\left\|\tfrac{1}{2}(y_n + y_m) - x\right\| \geq 2d$$

because M is convex , so that $\frac{1}{2}(y_n + y_m) \in M$.

Furthermore , we have $y_n - y_m = v_n - v_m$.

Hence by the parallelogram equality,

$$\|y_n - y_m\|^2 = \|v_n - v_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$$

$$\leq -(2d)^2 + 2\left({d_n}^2 + {d_m}^2\right),$$

since

$$d_n \longrightarrow d \quad where \;\; d_n = \|x - y_n\|$$

implies that $(y_n)$is Cauchy and converges;

M is complete , such that, $y_n \longrightarrow y \in M$.

Since $\|x - y\| \geq d$, $y \in M$. From

$$d_n \longrightarrow d \quad where \;\; d_n = \|x - y_n\| \, ,$$

$$\|x - y\| \leq \|x - y_n\| + \|y_n - y\| = d_n + \|y_n - y\| \longrightarrow d \, .$$

This shows that $\|x - y\| = d$.

2) Let $y \, , y_0 \in M$ both satisfy

$$\|x - y\| = d \;\; and \;\; \|x - y_0\| = d$$

and then y = y_0.

From theorem (2.1.1) ,

$$\|y - y_0\|^2 = \|(y - x) - (y_0 - x)\|^2$$

$$= 2\|y - x\|^2 + 2\|y_0 - x\|^2 - \|(y - x) + (y_0 - x)\|^2$$

$$= 2d^2 + 2d^2 - 2^2 \left\|\tfrac{1}{2}(y + y_0) - x\right\|^2.$$

On the right , $\frac{1}{2}(y + y_0) \in M$, so that

$$\left\|\tfrac{1}{2}(y + y_0) - x\right\| \geq d$$

implies that $2d^2 + 2d^2 - 4d^2 = 0$ is more than or equal the right- hand side. Hence

$$\|y - y_0\| \leq 0$$

so , $\|y - y_0\| \geq 0$, so that we must have equality, and y = y_0 .

Theorem (2.3.2) [20]

Suppose a Hilbert space is H , $M \subset H$ a non-empty closed and convex subset. Then for a point $m_y \in M$ the following assertions are equivalent;

1) $m_y = P_M(y);$
2) $Re\langle m - m_y, y - m_y\rangle \leq 0$ for all $m \in M$

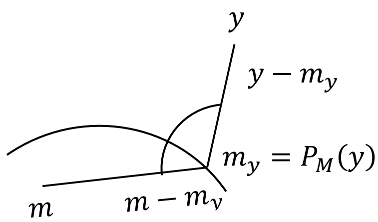

Fig.3. Projection onto a convex set

Proof

By translation we can assume that $m_y = 0$.

Assuming that

$m_y = 0 = P_M(y)$

By definition of $P_M(y)$, $\|y\| = \|y - 0\| = \inf_{m \in M} \|y - m\|$,

so $\|y\| \leq \|y - m\|$ for all $m \in M$. As $o, m \in M$ and M is convex we have

$$\|y\|^2 \leq \|y - tm\|^2 = \|y\|^2 + t^2\|m\|^2 - 2t\, Re\langle m, y\rangle$$

for all $m \in M$ and $t \in (0,1]$.

Hence

$$Re\langle m, y\rangle \leq \frac{t}{2}\|m\|^2$$

for all $m \in M$ and $t \in (0,1]$. If we fix $m \in M$ and let t go to zero , then $Re\langle m, y\rangle \leq 0$ as claimed . Now assume that $Re\langle m, y\rangle \leq 0$ for all $m \in M$ and that $0 \in M$.

We want to show that $0 = P_M(y)$ .If $m \in M$ we then have

$$\|y-m\|^2 = \|y\|^2 + \|m\|^2 - 2\, Re\langle y, m\rangle \geq \|y\|^2$$

since $Re\langle m, y\rangle \leq 0$ by assumption . As $0 \in M$ we conclude that

$$\|y\| = \inf_{m \in M} \|y - m\|,$$

so $0 = P_M(y)$ as claimed .

Every vector subspace M of a Hilbert space is obviously convex . If it is closed , then the above characterization of the projection can be applied .

The corollary also explains why P_M is called the orthogonal projection onto M .

Corollary (2.3.3) [20]

In Hilbert space H , M is a closed subspace. Then $m_x = P_M(x)$ iff $m_x \in M$ and $\langle x - m_x, m\rangle = 0$, $\forall m \in M$.

Moreover , $P_M : H \longrightarrow M$ is linear .

proof

By the above theorem $m_x = P_M(x)$ if and only if $Re\langle m_x - x, m - m_x\rangle \leq 0$ for all $m \in M$. Since M is a subspace $m + m_x \in M$ for all $\in M$,

So using $m + m_x$ instead of m we get that

$Re\langle m_x - x, (m + m_x) - m_x\rangle = Re\langle m_x - x, m\rangle \leq 0$

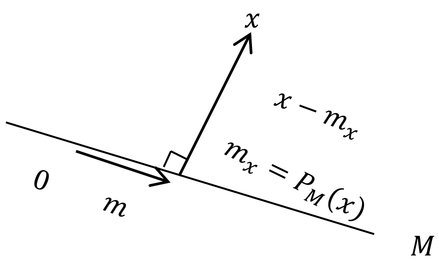

Fig.4. Projection onto a closed space

Replacing m by $-m$ we obtain

$$-Re\langle m_x - x, m\rangle = Re\langle m_x - x, -m\rangle \leq 0,$$

so we must have $Re\langle m_x - x, -m\rangle = 0 \;\; \forall\, m \in M$.

Similarly , replacing $m = \pm im$ if H is a complex Hilbert space we have

$$\pm Im\langle m_x - x, im\rangle = Re\langle m_x - x, \pm m\rangle \le 0,$$

Also $m\langle m_x - x, m\rangle = 0$.

So that $\langle m_x - x, m\rangle = 0$ for all $m \in M$ as claimed . It remains to show that P_M is linear . If $x, y \in H$ and $, \beta \in \mathbb{R}$, then by what we just proved

$$0 = \alpha\langle x - P_M(x)\,, \mathrm{m}\rangle + \beta\langle y - P_M(y)\,, \mathrm{m}\rangle$$

$$= \langle \alpha x + \beta y - \left(\alpha P_M(x) + \beta P_M(y)\right), m\rangle$$

for all $m \in M$. Hence a gain by what proved $P_M(\alpha x + \beta y) = \alpha P_M(x) + \beta P_M(y)$, showing that P_M is linear .

Lemma(2.3.4) [13]

If M is a complete subspace Y and $x \in X$ fixed. Assume that X is an inner product space. Then $z = x - Y$ is orthogonal to Y.

Proof

Let $z \perp Y$ were false, $y_1 \in Y$ would exist.

so that $\langle z, y_1\rangle = \beta \neq 0$.since , $y_1 \neq 0$

otherwise $\langle z, y_1\rangle = 0$,

furthermore , for any scalar α,

$$\|z - \alpha y_1\|^2 = \langle z - \alpha y_1\,, z - \alpha y_1\,\rangle$$

$$=\langle z, z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha[\langle y_1, z\rangle - \bar{\alpha}\langle y_1, y_1\rangle]$$

$$= \langle z, z\rangle - \bar{\alpha}\beta - \alpha\left[\bar{\beta} - \bar{\alpha}\langle y_1, y_1\rangle\right].$$

The expression in the brackets $\left[\bar{\beta} - \bar{\alpha}\langle y_1, y_1\rangle\right]$ is zero if we choose $\bar{\alpha} = \frac{\bar{\beta}}{\langle y_1, y_1\rangle}$.

We have $\|z\| = \|x - y\| = d$, so that our equation now yields

$$\|z-\alpha y_1\|^2 = \|z\|^2 - \frac{|\beta|^2}{\langle y_1, y_1\rangle} < d^2$$

but this is impossible because we have

$z-\alpha y_1 = x - y_2$ where $y_2 = y + \alpha y_1 \in Y$,

So that $\|z-\alpha y_1\| > d$ by the definition of d .

Hence $\langle z, y_1\rangle = \beta \neq 0$ cannot hold and, the Lemma is proved .

2.4 Orthogonal Complements and Direct Sums.

Definition (2.4.1)

If subspaces Z and W of a vector space X then X is called the direct sum ,such that

$$X = Z \oplus W,$$

if any $x \in X$ is represented a unique

$$x = z + w \ , z \in Z \ , w \in W.$$

When this occurs, Z and W are called complementary pairs of subspaces in X and vice versa and W is said to the algebraic complement of Z in X.

The main interest concerns representations of H, in the case of general Hilbert space H

The **orthogonal complement** is a direct sum of a closed subspace Y such that

$$Z^{\perp} = \{w \in H \, ; w \perp Z\},$$

It`s the set of all vectors orthogonal to Z .

Theorem (2.4.1) [11]

If Z is each closed subspace of Hilbert space H. Then

$$H = Z \oplus W$$

Proof

Since Z is closed and Z, H are complete.

Since Z is convex , for every $x \in H$ there is a $z \in Z$ such that

$$x = z + w \quad w \in W = Z^{\perp}$$

To prove uniqueness , suppose that

$$x = z + w = z_1 + w_1$$

where $z , z_1 \in Z$ and $w , w_1 \in W$.

Then $z - z_1 = w_1 - w$.

Since $z - z_1 \in Z$ whereas $w_1 - w \in W = Z^{\perp}$,

and $z - z_1 \in Z \cap Z^{\perp} = \{0\}$.

This implies $z = z_1$. Hence also $w_1 = w$.

Theorem (2.4.2) [1]

Let linear operator be T from vector space Y into vector space X. Then

$$\dim Y = \dim \ker T + \dim R(T).$$

Proof

We assume that B is completed for $kerT$ space in Y
imply that $Y = \ker T \oplus B$. Then

$$\dim Y = \dim \ker T + \dim B.$$

Such that

$$\dim B = \dim R(T)$$

Implies

$$\dim Y = \dim \ker T + \dim R(T) \ .$$

Since P is a bounded linear operator.Where P maps H onto Y, and maps Y onto itself,

$$z = Y^{\perp}\ onto\ \ \{0\},$$

and P is **idemptent** , that is P^2=P;

hence , for all $x \in H, P^2x$=$P(Px) = Px$.

Lemma(2.4.3) [13]

If H is a Hilbert space and Z be a closed subspace of H ,then $Z = Z^{\perp\perp}$

Proof

we have $Z \subset Z^{\perp\perp}$ because $y \in Z$

implies $y \perp Z^{\perp}$ and $y \in (Z^{\perp\perp})$

Now. Let $\in Z^{\perp\perp}$. Then

$y = z + w\ \ where\ \ z \in\ Z \subset Z^{\perp\perp}$.

Since $y \in Z^{\perp\perp}\ becuse\ \ Z^{\perp\perp}$ is a vector space,

we have $w = y - z \in Z^{\perp\perp}$,

hence, $w \perp Z^{\perp}$. But $y \in Z^{\perp}$. Together $w \perp w$,

so that w=0, y=z, thus, $y \in Z$.

Since $y \in Z^{\perp\perp}$,this proves $Z^{\perp\perp} \subset Z$.

Theorem (2.4.4) [15]

The orthogonal space of subsets $B \subset Y$,

$$B^{\perp} = \{y \in Y\ ;\ \langle y, b\rangle = 0\ , \forall\ b \in B\},$$

satisfy

1) $B \cap B^{\perp} \subseteq \{0\}$,
2) $B^{\perp}$ be a closed subspace of Y.

Proof

1) Let a vector $b \in B$ is in $B^{\perp}$, then its orthogonal to all vectors in B ,

including itself , $\langle b, b\rangle = 0$, $so\ b = 0$.

2) Let y and z are in $B^{\perp}$ and $b \in B$, then

$$\langle \alpha y, b\rangle = \bar{\alpha}\langle y, b\rangle = 0 \ ,$$

$$\langle y + z \ , b\rangle = \langle y, b\rangle + \langle z, b\rangle = 0,$$

So $\alpha y \ , y + z \in B^{\perp}$. If $y_n \in B^{\perp}$ and $y_n \longrightarrow y$, then

$$0 = \langle y_n \ , b\rangle \longrightarrow \langle y, b\rangle \ , and \ \ y \in B^{\perp}.$$

Theorem (2.4.5) [13]

$S^{\perp} = \{0\}$ iff the span of S is dense in H for each subset $S \neq \emptyset$ of a Hilbert space H .

Proof

assume $S^{\perp} = \{0\}$. Let $z \perp V$, then $z \perp S$, hence $z \in S^{\perp}$ and $z = 0$. Thus $V^{\perp} = \{0\}$. Such that V is subspace of H, we obtain $\bar{V} = H$ with $Z = \bar{V}$.

Conversely, If z $\in S^{\perp}$ and suppose $V = span\ S$ is dense in H .
Then $\in \bar{V} = H$.

This implies the sequence (z_n),which is existed in V such that $z_n \longrightarrow z$.
So that $z \in S^{\perp}$ and $S^{\perp} \perp V$,

since $\langle z_n, z\rangle = 0$.

By Lemma (2.2.4) implies that

$\langle z_n \ , z\rangle \longrightarrow \langle z, z\rangle$.

Together, $\langle z, z\rangle = \|z\|^2 = 0$,

thus $z = 0$. Since $z \in S^{\perp}$, hence $S^{\perp} = \{0\}$.

Theorem(2.4.6) [15]

Let M is a closed vector subspace of a Hilbert space H ,then $w \in M$ is the closest point w to $z \in H$ if and only , $z - w \in M^{\perp}$.

The map $p: z \longrightarrow w$ is a continuous, orthogonal projection with $Imp = M$

orthogonal to ker $P = M^{\perp}$, so $H = M \oplus M^{\perp}$

Proof

1) If a be any nonzero point of M and let $c \coloneqq z - (w + \alpha b)$ $where\ \alpha$ is chosen

So that $\perp c$, that is , $\alpha \coloneqq \langle b, z - w\rangle / \|b\|^2$.

By(Pythagoras) ,we get

$$\|z - w\|^2 = \|c + \alpha b\|^2 = \|c\|^2 + \|\alpha b\|^2 \geq \|c\|^2$$

Making $w + \alpha b$ even closer to x than the closest point y , unless

$$\alpha = 0 \ , \langle b, z - w\rangle = 0.$$

Since b is arbitrary , thus $(z - w) \perp M$.

Conversely , let $(z - w) \perp \grave{b}$ for each $\grave{b} \in M$, then $(z - w) \perp (\grave{b} - w)$ and (Pythagoras) implies

$$\|z - \grave{b}\|^2 = \|z - w\|^2 + \|w - \grave{b}\|^2 ,$$

So that $\|z - w\| \leq \|z - \grave{b}\|$, let w the closest point in M to z.

2) For any $z \in H$,$P(z)$ is that unique vector in M such that

$$z - P(z) \in M^{\perp}.$$

This characteristic property has the following

P is linear since $(z + w) - (pz + pw)$=$(z - pz) + (w - pw) \in M^{\perp}$,

$$pz + pw \in M, hence\ \ p(z + w) = pz + pw .$$

Similarly, $p(\alpha z) = \alpha pz$.

The closest point in M to $b \in M$ is a itself, $pb = b\ , so\ Imp = M$.

Since $pz \in M,$ it also follows that $p^2 z = pz,$ and $p^2 = p$.

When $\in M^{\perp}$, then $z - 0 \in M^{\perp}$ and $0 \in M$ so $pz = 0$.

As $pz = 0$ implies $z = z - pz \in M^{\perp}$, this just itself $\ker p = M^{\perp}$.

since $\|z\|^2 = \|z - pz\|^2 + \|pz\|^2$,P is continuous

thus $\|pz\| \leq \|z\|$.

hence $H = Im\ p \oplus ker\ p = M \oplus M^{\perp}$,since any vector can be decomposed as

$z = pz + (z - pz) \quad , and\ \ M \cap M^{\perp} = \{0\}$.

2.5 Orthonormal sets and Sequences

In this section we will show some definition and important theorem .

Definition (2.5.1)

Let X be an orthogonal set M in the inner product space X is a subset of $M \subset X$ with pairwise orthogonal elements. For all $y, z \in M$,

$$\langle y, z \rangle = \begin{cases} 0 & if\ \ y \neq z \\ 1 & if\ \ y = z. \end{cases}$$

making an orthonormal set $M \subset X$ an orthogonal set in X with elements of norm 1.

An indexed set or family, (y_α), $\alpha \in I$; is **called orthogonal** if $y_\alpha \perp y_\beta\ \ for\ all\ \ \alpha, \beta \in I \quad \alpha \neq \beta$ is an orthogonal or orthonormal set M is countable, the sequence (y_n), and it is an orthogonal or orthonormal sequence.

If family is orthogonal and all y_α have norm 1 ,then it is called orthonormal

for all $\alpha, \beta \in I$ hence

$$\langle y_\alpha, y_\beta \rangle = S_{\alpha\beta} = \begin{cases} 0 & if\ \ \alpha \neq \beta, \\ 1 & if\ \ \alpha = \beta. \end{cases}$$

Here , $S_{\alpha\beta}$ is the kronecker delta .

Next we give some examples.

Examples(2.5.1)

1) Space l^2 **.** In this space, (e_n) is an orthonormal sequence when $e_n = (S_{nj})$ has the n^{th} element 1 and all others zero
2) Space $\mathbb{R}^3$. $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ is three unit vectors in this space.

Theorem (2.5.1) [13]

If an orthonormal set then is linearly independent .

Proof

If $\{e_1 , \dots , e_n\}$ is orthonormal and consider the equation

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0.$$

$$\sum \alpha_i e_i = 0$$

Multiplication by a fixed e_j gives

$$\langle \sum_{j=1}^{n} \alpha_i e_i \ , e_j \rangle = \sum_{j=1}^{n} \alpha_i \langle e_i , e_j \rangle = \alpha_j \langle e_i , e_j \rangle = \alpha_j = 0 \ ; \ \forall j = 1, \dots , n.$$

The following theorem, Gram Schmidt, which proves shows how to transform the linear independent sets

Into orthogonal sets, and to transform these sets into orthonormal sets in the inner product spaces**.**

Theorem (2.5.2) [19]

Let X inner product space

If $\{y_n\}_{n=1}^{\infty}$ linearly independent sequence in X, then there sequence $\{z_n\}_{n=1}^{\infty}$ from orthonormal vector such that

$$span\{y_n\} = span\{z_n\}.$$

Proof

Note that $y_n \neq 0$ for any n. Because set $\{y_n\}$ is linearly independent.

Let

$$z_1 = \frac{w_1}{\|w_1\|} \ , w_1 = y_1$$

$$z_2 = \frac{w_2}{\|w_2\|} \ , w_2 = y_2 \ - \langle y_2, z_1 \rangle z_1$$

$$\vdots$$

$$z_{n+1} = \frac{w_{n+1}}{\|w_{n+1}\|} \ , w_{n+1} = \ y_{n+1} - \sum_{k=1}^{n} \langle y_{n+1} \, , z_k \rangle \, z_k$$

note that $z_1 \perp w_2$, and also w_{n+1} orthogonal with for every $z_1 \, , z_2 \, , \dots , z_n$

note that $\{z_n\}_{n=1}^{\infty}$ is orthonormal,

and z_n is linear combination for element $y_1 \, , y_2 \, , \dots$

Conversely , hence

$$span\{y_n\} = span\{w_n\}$$

The following result determined the linear combination for the elements of orthonormal sequences .

Theorem (2.5.3) Bessel inequality [15]

If the orthonormal sequence$\{y_n\}_{n=1}^{\infty}$ in an inner product space X . Then $\forall y \in X$

$$\sum_{j=1}^{\infty} \left|\langle y \, , y_j \rangle\right|^2 \leq \|y\|^2$$

Proof

We have

$$\left|\langle y \, , y_j \rangle\right|^2 \leq \ |\langle y \, , y_1 \rangle|^2 \leq |\langle y \, , y_2 \rangle|^2 \ \leq \cdots$$

This show that sequence $\left\{\sum_{j=1}^{n}\left|\langle y\,,y_j\rangle\right|^2\right\}_{n=1}^{\infty}$ as bounded increasing series,

then

$\sum_{j=1}^{\infty}\left|\langle y\,,y_j\rangle\right|^2$ is convergent.

Now

$$0 \le \langle y - \sum_{j=1}^{n}\langle y, y_j\rangle\, y_j\,, y - \sum_{j=1}^{n}\langle y, y_j\rangle\, y_j\,\rangle$$

$$= \|y\|^2 - \sum_{j=1}^{n}\left|\langle y\,,y_j\rangle\right|^2$$

hence

$$\sum_{j=1}^{n}\left|\langle y\,,y_j\rangle\right|^2 \le \|y\|^2$$

As $n \longrightarrow \infty$ we obtain

$$\sum_{j=1}^{\infty}\left|\langle y\,,y_j\rangle\right|^2 \le \|y\|^2.$$

Definition (2.5.2)

Let H is a Hilbert space and $\{x_i\}$ be an orthonormal sequence in H, then for every $x \in H$,

The **Fourier coefficient of x** is $\langle x\,,x_i\rangle$ and $\sum_{i=1}^{\infty}\langle x, x_i\rangle\, x_i$ is

Fourier series with respect to $\{x_i\}$.

Definition (2.5.3)

In normed space V , $\{x_n\}$ be a sequence, say that $\sum_{n=1}^{\infty} x_n$ converges and has Sum $x(\sum_{n=1}^{\infty} x_n = x)$ if $\sum_{n=1}^{N} x_n \longrightarrow x$ as $\mathbb{N} \longrightarrow \infty$.

$\|x - \sum_{n-1}^{N} x_n\| \longrightarrow 0 \quad as \quad \mathbb{N} \longrightarrow \infty$.

Theorem(2.5.4) [11]

If $\{\alpha_k\}$ is a sequence in $\mathbb{C}$,and if $\{e_k\}$ be an orthonormal sequence in Hilbert space H .Then $\sum_{k=1}^{\infty} \alpha_k e_k$ converges in H iff $\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$.

Proof

($\Longrightarrow$)Let $x = \sum_{k=1}^{\infty} \alpha_k e_k$ and $x_k = \sum_{k=1}^{N} \alpha_k e_k$, then $\langle x_N , e_k\rangle = \alpha_k$ for $k < N$.

and taking $N \longrightarrow \infty$, gives $\langle x , e_k\rangle = \alpha_k$.Then by Bessel inquality

$$\sum_{k=1}^{\infty} |\alpha_k|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2 < \infty.$$

($\Longleftarrow$)

Assume that $\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$, and let $x_k = \sum_{k=1}^{N} \alpha_k e_k$. Then

$$\left\|x_{N+p} - x_N\right\|^2 = \left\|\sum_{k=N+1}^{N+p} \alpha_k e_k\right\|^2 = \sum_{k=N+1}^{N+p} \|\alpha_k e_k\|^2$$

$$= \sum_{k=N+1}^{N+p} |\alpha_k|^2 \longrightarrow 0 \;\; as \;\; N \longrightarrow \infty.$$

Therefore $\{x_N\}$ is Cauchy, and it converges in H .

Theorem (2.5.5) [13]

If H a Hilbert space and (e_k) is an orthonormal sequence in H .Then

let $\sum_{k=1}^{\infty} \alpha_k e_k$ converges , then the cofficients α_k are the Fourier cofficients

$\langle x, e_k\rangle$, when x denotes the sum of $\sum_{k=1}^{\infty} \alpha_k e_k$; hence , $\sum_{k=1}^{\infty} \alpha_k e_k$

can be written

$$x = \sum_{k=1}^{\infty} \langle x, e_k\rangle \, e_k.$$

Proof

By using the orthonormality and taking the inner product of s_n and e_j and, we obtain

$\langle s_n, e_j\rangle = \alpha_j$ for $j = 1, \dots, k \qquad (k \le n \ and\ fixed)$.

By assumption , $s_n \longrightarrow x$. By Lemma (2.2.4),the inner product is continuous

$$\alpha_j = \langle s_n, e_j\rangle \longrightarrow \langle x, e_j\rangle \qquad\qquad (j \le k).$$

and take $k(\le n)$ as large as we please because n $\longrightarrow \infty$,

hence

$$\alpha_j = \langle x, e_j\rangle \quad for\ every\ \ j = 1,2, \dots.$$

Lemma (2.5.6) [13]

for any $x \in X$ if X is an inner product space can have at most countably many nonzero Fourier coefficients $\langle x, e_k\rangle$ with respect to an orthonormal family$(e_k), k \in I\ , in\ X$.

proof

We can associate a series similar to $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle\, e_k$ for any fixed $x \in H$

$\sum_{k\in I}\langle x, e_k\rangle\, e_k$ and we can arrange the e_k with $\langle x, e_k\rangle \ne 0$ in a sequence $(e_1, e_2\ , \dots)$, so that $\sum_{k\in I}\langle x, e_k\rangle\, e_k$ takes the form $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle\, e_k$. convergence by Theorem (2.5.5) .

If (w_m) is a rearrangement of (e_n). In order for the corresponding terms of the two sequences to be equal , since there exists a bijective mapping $n \longmapsto m(n)$ of N onto itself . Thus , $w_{m(n)} = e_n$.

We set

$$\alpha_n = \langle x, e_n\rangle \quad , \qquad \beta_m = \langle x, w_m\rangle$$

and

$$x_1 = \sum_{n=1}^{\infty} \alpha_n e_n \quad , \quad x_2 = \sum_{m=1}^{\infty} \beta_m w_m .$$

Then by Theorem (2.5.5),

$$\alpha_n = \langle x, e_n \rangle = \langle x_1, e_n \rangle \quad , \quad \beta_m = \langle x, w_m \rangle = \langle x_2, w_m \rangle .$$

Since $e_n = w_{m(n)}$,we thus obtain

$$\langle x_1 - x_2 , e_n \rangle = \langle x_1, e_n \rangle - \langle x_2 , w_{m(n)} \rangle$$

$$= \langle x, e_n \rangle - \langle x , w_{m(n)} \rangle = 0$$

and similarly $\langle x_1 - x_2 , w_m \rangle = 0$. This implies

$$\|x_1 - x_2\|^2 = \langle x_1 - x_2 , \sum \alpha_n e_n - \sum \beta_m w_m \rangle$$

$$= \sum \overline{\alpha_n} \langle x_1 - x_2, e_n \rangle - \sum \overline{\beta_m} \langle x_1 - x_2, w_m \rangle = 0 .$$

Consequently , $x_1 - x_2 = 0 \quad and \; x_1 = x_2$.

Since the rearrangement (w_m) of (e_n) was arbitrary .

2.6 Total Orthonormal Setes and Sequences.

Definition(2.6.1)

An orthonormal set A in an inner product space X cannot be expanded to a larger orthonormal set and X is maximal if the only point in X which is orthogonal to every $y \in A$ is 0 . Also ,A is total if its span is dense in X ; in this case , every $y \in X$ so that as $y = \sum_{e \in A} \langle y, e \rangle \, e$,and A is said to be **an orthonormal basis of X.**

$\overline{span\ A} = X$ if and only if A is total in X.

Theorem (2.6.1) [19]

If X is an inner product space and B be a subset of X. Then

1) Let B be total in X , there is no nonzero $x \in X$ that is orthogonal to each element of B;

$$x \perp B \implies x = 0$$

2) Let X be complete, then the totality of B in X is sufficient satisfies that condition.

Proof

1) If X is considered a subspace of H and H is the completion of X, then X is dense in H. Considering that B is total in X, span B is dense in X and hence dense in H. Theorem (2.4.5) now implies that the orthogonal complement of B in H is $\{0\}$. If $x \in X$ and $x \perp B$, *then* $x = 0$.
2) Let B satisfies $x \perp B \implies x = 0$ and X be a Hilbert space, hence $B^{\perp} = \{0\}$, then Theorem (2.4.5) implies that B is total in X .

Theorem (2.6.2) [13]

If a Hilbert space is H .Then

1) Assuming that H is separable, each orthonormal set in H can be countable.
2) Let H contains an orthonormal sequence that is total in H, then H can be separable.

Proof

1) Let M any orthonormal set and B any dense set in H. Then

any two distinct elements x and y of M have distance $\sqrt{2}$ Thus

$$\|x - y\|^2 = 2 .$$

Since N_x of x and N_y of y are spherical neighborhoods radius $\sqrt{2}/3$ disjoint.

There is a $b_1 \in B$ *in* N_x and a $b_2 \in B$ *in* N_y and $b_1 \neq b_2$,since B is dense in H

since $N_x \cap N_y = \emptyset$. Because of this, if M were uncountable, there would be an infinite number of these pairwise disjoint spherical neighborhoods ($\forall x \in M$

one of them), making B uncountable. Given that B might be any dense set, separability is contradicted since H cannot contain a dense set that is countable. This leads us to the conclusion that M must be countable.

2) Assuming that (e_k) is a total orthonormal sequence in H the set of all possible linear combinations

$$\alpha_1{}^{(n)} e_1 + \alpha_2{}^{(n)} e_2 + \cdots + \alpha_n{}^{(n)} e_n \qquad n = 1,2, \dots$$

where $\alpha_k{}^{(n)} = a_k{}^{(n)} + i b_k{}^{(n)}$ and $a_k{}^{(n)}$ and $b_k{}^{(n)}$ are rational

(and $b_k{}^{(n)} = 0 \; if \;\; H \; is \; real$). A is countable .

By showing that for every $z \in H \;\; and \; \epsilon > 0$ there is a $v \in A$ such that $\|z - v\| < \epsilon$ to prove that A is dense in H.

There is an n such that $Y_n = span \; \{e_1 , \dots e_n\}$, So that (e_k) is total in H

We obtain

$\|z - y\| < \epsilon/2$ for the orthogonal projection $y \in Y_n \; and \; z \; on \; Y_n$.

Now

$$Y = \sum_{k=1}^{n} \langle z, e_k \rangle \, e_k.$$

Hence

$$\left\| z - \sum_{k=1}^{n} \langle z, e_k \rangle \, e_k \right\| < \epsilon/2$$

The rationals in $\mathbb{R}$ are dense , for every $\langle z, e_k \rangle$ exist $\alpha_k{}^{(n)}$ such that

$$\left\| \sum_{k=1}^{n} \left[\langle z, e_k \rangle - \alpha_k{}^{(n)} \right] e_k \right\| < \epsilon/2.$$

Hence $v \in A$ defined by

$$v = \sum_{k=1}^{n} \alpha_k{}^{(n)}\, e_k$$

$satisfies$

$$\|z - v\| = \left\| z - \sum \alpha_k{}^{(n)} e_k \right\|$$

$$\leq \left\| z - \sum \langle z, e_k \rangle e_k \right\| + \left\| \sum \langle z, e_k \rangle e_k - \sum \alpha_k{}^{(n)} e_k \right\|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

$This\ proves\ that\ A\ is\ dence\ in\ H,\ \ since\ A\ countable\ and\ H\ is\ separable\ .$

Chapter Three

3 Linear Operators on Hilbert Spaces

We have discussed basic concept of linear operators already. and this section here we want to prove some quiteun expected results on bounded linear operator on Hilbert spaces.

3.1 Linear Functionals on Hilbert Spaces.

Riez`s Theorem (3.1.1) [13]

Let H a Hilbert space and f be bounded linear functional on H then H is equivalent to the inner product,

$$f(x) = \langle y, w\rangle,$$

When w depends on f, f determines it uniquely, and its norm is

$$\|w\| = \|f\|$$

Proof

If $f = 0 \,.Then\, f$ has a representation if we take $= 0$.

Let $\neq 0$, this implies $w \neq 0$, thus otherwise $f = 0$.

And $\langle y, w\rangle = 0$, $\forall\, y$, where $f(y) = 0$, such that, for every y in the null space $N(f)$ of f . Thus $w \perp N(f)$. The implication is that we take into account $N(f)$and its orthogonal complement $N(f)^{\perp}$.

$N(f)$ be closed and a vector space. $N(f) \neq H$ is implied by $f \neq 0$,

thus $N(f)^{\perp} \neq \{0\}$. Hence contains a $w_0 \neq 0$. we set

$$u = f(y)w_0 - f(w_0)y,$$

where $y \in H$ is arbitrary Applying f, hence

$$f(u) = f(y)f(w_0) - f(w_0)f(y) = 0$$

thus $\in N(f)$. since $w_0 \perp N(f)$,

we obtain

$$0 = \langle u, w_0\rangle = \langle f(y)w_0 - f(w_0)y\,, w_0\rangle$$

$$= f(y)\langle w_0 , w_0\rangle - f(w_0)\langle y, w_0\rangle.$$

Noting that $\langle w_0 , w_0\rangle = \|w_0\|^2 \neq 0$, we can solve for $f(y)$.

hence

$$f(y) = \frac{f(w_0)}{\langle w_0 , w_0\rangle}\langle y, w_0\rangle.$$

Since it was arbitrary, this can be expressed as $f(y) = \langle y, w\rangle$ where

$w = \frac{\overline{f(w_0)}}{\langle w_0 ,w_0\rangle} w_0$.

We prove w in $f(y) = \langle y, w\rangle$ is unique .

Suppose that for $\in H$,

$$f(y) = \langle y, w_1\rangle = \langle y, w_2\rangle.$$

Then $\langle y, w_1 - w_2\rangle = 0$ for every y . choosing $= w_1 - w_2$,

we have

$$\langle y, w_1 - w_2\rangle = \langle w_1 - w_2 , w_1 - w_2\rangle = \|w_1 - w_2\|^2 = 0.$$

Hence $w_1 - w_2 = 0$, so that $w_1 = w_2$, the uniqueness .

Now

Let $f = 0$, then $w = 0 \ \ and \ \ \|w\| = \|f\| \ \ hold.$

If $f \neq 0$. Then $w \neq 0 \ with \ y = w$ and we obtain

$$\|w\|^2 = \langle w, w\rangle = f(w) \leq \|f\|\|w\|.$$

Divided by $\|w\| \neq 0$, the result is $\|w\| \leq \|f\|. \longrightarrow (1)$.

From the Schwarz inquality we see that

$$|f(y)| = |\langle y, w\rangle| \leq \|y\|\|w\|.$$

This implies

$$\|f\| = \sup_{\|y\|=1} |\langle y, w\rangle| \leq \|w\|.$$

$$\|f\| \leq \|w\| \longrightarrow (2)$$

By (1) and (2) we obtain

$$\|f\| = \|w\|$$

3.2 Sesquilinear Form.

We showing some definition and theorem in this section.

Definition (3.2.1)

If $k = (\mathbb{R}\ or\ \mathbb{C})$ and Y ,Z are vector space on field k .Then a sesquilinear form h on $Y \times Z$ be a mapping

$$h: Y \times Z \longrightarrow k$$

$\forall\ y, y_1, y_2 \in Y\ and\ \forall\ z, z_1, z_2 \in Z$ and all scalars α, β,

1) $h(y_1 + y_2, z) = h(y_1, z) + h(y_2, z)$
2) $h(y, z_1 + z_2) = h(y, z_1) + h(y, z_2)$
3) $h(\alpha y, z) = \alpha h(y, z)$
4) $h(y, \beta z) = \bar{\beta} h(y, z)$.

So that in the first argument, h is linear, while in the second, it is conjugate linear.

let Y , Z both is real($K = \mathbb{R}$), thus

$$h(y, \beta z) = \beta h(y, z)$$

Definition (3.2.2)

If Y is a vector space on the field K .A Hermitian forms h on $Y \times Y$ is a mapping

$$h: Y \times Y \longrightarrow K$$

for every $y, z, w \in Y\ and\ \alpha \in K$,

$$h(y+z,w) = h(y,w) + h(z,w)$$

$$h(\alpha y,z) = \alpha h(y,z)$$

$$h(y,z) = \overline{h(y,z)}$$

If a sesquilinear form h on Y has the property following, it is said to be nondegenerate .

Let $y \in Y$ be $h(y,z) = 0 \ \forall z \in Y$, then $y = 0$;

If $z \in Y$ is $h(y,z) = 0 \ \forall y \in Y$, then $z = 0$.

In particular , forms are Hermitian positive definite sesquilinear.

It is clear that they are nondegenerate. **Nonegative sesquilinear forms** are sequilinear forms that satisfy the weaker requirement, which is for any $y \in Y$, $y \neq 0$, $h(y,y) \geq 0$.

Theorem (3.2.1) [19]

If the complex vector space X and nonegative sesquilinear form is h on X. Then,

$$|h(x,y)|^2 \leq h(x,x)h(y,y) \quad for\ all\ x,y \in X.$$

Proof

If $h(x,y) = 0$, the inquality is , true .

Suppose $h(x,y) \neq 0$.

Such that α , β any arbitrary complex numbers ,we have

$$0 \leq h(\alpha x + \beta y, \alpha x + \beta y)$$

$$= \alpha\bar{\alpha}h(x,x) + \alpha\bar{\beta}h(x,y) + \bar{\alpha}\beta h(y,x) + \beta\bar{\beta}h(y,y)$$

we have

$$\beta h(y,x) = \bar{\alpha}\beta\overline{h(x,y)}$$

Since h is nonnegative . Now

If $\alpha = t$ is real and set

$$\beta = h(x,y)/|h(x,y)|.$$

Then,

$$\beta h(y,y) = |h(x,y)| \text{ and } \beta\bar{\beta} = 1.$$

Hence,

$$0 \leq t^2 h(x,x) + 2t|h(x,y)| + h(y,y)$$

T is an arbitrary real number t . Hence ,the discriminant

$$4|h(x,y)|^2 - 4h(x,x)h(y,y) \geq 0,$$

Definition (3.2.3)

If a Hilbert space is H. If there is a positive constant M such that $|h(x,y)| \leq M\|x\|\|y\|$ for all $x, y \in H$, then the sesquilinear form h is **said to be bounded**.

The norm of h is defined by

$$\|h\| = \sup_{\|x\|=\|y\|=1} |h(x,y)| = \sup_{\substack{x\in H, y\in H \\ x\neq 0\neq y}} \frac{|h(x,y)|}{\|x\|\|y\|}.$$

Examples (3.2.1)

1) Assuming H is a Hilbert space, part (1) of the theorem (2.2.1) states that the sesquilinear form $h: H \times H \longrightarrow \mathbb{C}$ defined by $h(y,z) = (y,z)$ is bounded . $\|h\| = 1$. $Indeed$, $|h(y,z)| = |(y,z)| \leq \|y\|\|z\|$,

$and\ so$, $\|h\| \leq 1$.$For\ z = y$, $|h(y,z)| = |(y,y)| = \|y\|^2 = 1\ if\ \|y\| = 1$.

2) Let $T: H \longrightarrow H$ be a bounded linear operator ,then

$h(z,w) = (Tz,w)$ is a bounded sesquilinear forms with $\|h\| = \|T\|$.
Indeed , $\forall\ z, w \in H$, $\|z\| = \|w\| = 1$

$$|h(z,w)| = |(Tz,w)| \leq \|Tz\|\|w\| \leq \|T\|.$$

Hence,

$$\|h\| \leq \|T\|.$$

On the other hand ,for $w = Tz$,

$$\|h\| \geq \frac{|h(z,Tz)|}{\|z\|\|Tz\|} = \frac{\|Tz\|^2}{\|z\|\|Tz\|} = \frac{\|Tz\|}{\|z\|},$$

which implies

$$\|h\| \geq \|T\|$$

Theorem(3.2.2) [13]

If H_1 , H_2 are Hilbert spaces and

$$h: H_1 \times H_2 \longrightarrow K$$

a bounded sesquilinear form. The a representation of h is then

$$h(y,w) = \langle Sy, w\rangle$$

where a linear operator $S: H_1 \longrightarrow H_2$ is bounded.

S have norm $\|S\| = \|h\|$and be uniquely .

Proof

If $\overline{h(y,w)}$ is linear in , we keep y fixed.There is v so that

$$\overline{h(y,w)} = \langle w, v\rangle$$

Hence

$$h(y,w) = \langle v, w\rangle.$$

here $v \in H_2$ is unique but , depends on our fixed $y \in H_1$. Defines an operator

$S: H_1 \longrightarrow H_2$ given by $v = Sy$.

Thus

$$h(y,w) = \langle Sy, w\rangle$$

Prove that S is linear.

$$\langle S(\alpha y_1 + \beta y_2), w\rangle = h(\alpha y_1 + \beta y_2, w)$$

$$= \alpha h(y_1, w) + \beta h(y_2, w)$$

$$= \alpha\langle Sy_1, w\rangle + \beta\langle Sy_2, w\rangle$$

$$= \langle \alpha Sy_1 + \beta Sy_2, w\rangle$$

for all w in H_2 , so that

$$S(\alpha y_1 + \beta y_2) = \alpha Sy_1 + \beta Sy_2$$

S is bounded. In case $S = 0$,we have

$$\|h\| = \sup_{\substack{y\neq 0\\ w\neq 0}} \frac{|\langle Sy, w\rangle|}{\|y\|\|w\|} \geq \sup_{\substack{y\neq 0\\ S_w\neq 0}} \frac{|\langle Sy, Sy\rangle|}{\|y\|\|Sy\|} = \sup_{y\neq 0} \frac{\|Sy\|}{\|y\|} = \|S\|.$$

This proves boundedness. Moreover , $\|h\| \geq \|S\|$.

Now

$$\|h\| = \sup_{\substack{y\neq 0\\ w\neq 0}} \frac{|\langle Sy, w\rangle|}{\|y\|\|w\|} \leq \sup_{y\neq 0} \frac{\|Sy\|\|w\|}{\|y\|\|w\|} = \|S\|.$$

S is unique. For every $y \in H_1$ and $w \in H_2$,we have the following thanks to the linear operator$T: H_1 \longrightarrow H_2$ such that

$$h(y, w) = \langle Sy, w\rangle = \langle Ty, w\rangle,$$

we see that $\langle Sy - Ty, w\rangle = 0$.

so that $Sy = Ty$ for all $y \in H_1$.

Hence $S = T$

Corollary (3.2.3)[19]

Let S is the bounded sesquilinear functional satisfies the condition

$$|S(z, y)| = |S(y, z)|\ , z, y \in H,$$

then

$$\|S\| = \sup_{\substack{z\in H \\ \|z\|\neq 0}} \frac{|S(z,z)|}{\|z\|^2}$$

Proof:

It is evident that the supermum in issue is a potential value of M that satisfies

$$|S(z,z)| \leq M\|z\|^2$$

It follows that

$$\|S\| \leq \sup_{\substack{z\in H \\ \|z\|\neq 0}} \frac{|S(z,z)|}{\|z\|^2};$$

but one the other hand,

$$\sup_{\substack{z\in H \\ \|z\|\neq 0}} \frac{|S(z,z)|}{\|z\|^2} \leq \sup_{\substack{z\in H, y\in H \\ z\neq 0\neq y}} \frac{|S(z,y)|}{\|z\|\|y\|} = \|h\|.$$

3.3 Hilbert-Adjoint Operator

"Bilinear form research on a Hilbert space when H is a Hilbert space, B(H) is called a specific Banach algebra exists. A canonical bijection $T \longrightarrow T^{*}$* with appealing algebraic features is admissible in the algebra B(H) of bounded linear operators on H. Moreover, several features of T can be explored using the self adjoint operator T^{*} "[18]

Definition (3.3.1)

When $H_1, H_2$are Hilbert spaces, $T: H_1 \longrightarrow H_2$ is a bounded linear operator. For $\langle Tx, y\rangle = \langle x, T^{*}y\rangle \;\forall x \in H_1, y \in H_2$,the Hilbert adjoint operator T* of T is the operator $T^{*}: H_2 \longrightarrow H_1$.

Theorem (3.3.1) [13]

If T^{*} is Hilbert-adjoint operator of T Def (3.3.1) exists , be a bounded linear operator and unique with norm

$$\|T^*\| = \|T\|.$$

Proof

The formula

$$B(y,x) = (y,Tx)$$

since the inner product of sesquilinear and T is linear, defines a sesquilinear form on $H_2 \times H_1$. The formula's conjugate linearity is seen from

$$B(y,\alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2)\rangle$$

$$= \langle y, \alpha T x_1 + \beta T x_2\rangle$$

$$= \bar{\alpha}\langle y, T x_1\rangle + \bar{\beta}\langle y, T x_2\rangle$$

$$= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2).$$

B is bounded .

$$|B(y,x)| = |\langle y, Tx\rangle| \le \|y\|\|Tx\| \le \|T\|\|x\|\|y\|.$$

Implies

$$\|B\| \le \|T\|.$$

we obtain $\|B\| \ge \|T\|$ from .

$$\|B\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle y, Tx\rangle|}{\|y\|\|x\|} \ge \sup_{\substack{x\neq 0\\ Tx\neq 0}} \frac{|\langle Tx, Tx\rangle|}{\|Tx\|\|x\|} = \|T\|.$$

$$\|B\| = \|T\|.$$

By Theorem(3.2.2) ,we obtain

$$B(y,x) = \langle T^* y, x\rangle,$$

and since $T^*: H_2 \longrightarrow H_1$ is a bounded linear operator that can a uniquely be computed once and whose norm is

$$\|T^*\| = \|B\| = \|T\|.$$

Thus

$$\|T^*\| = \|T\|.$$

Also $\langle y, Tx\rangle = \langle T^*y, x\rangle$ by comparing $B(y,x) = \langle y, Tx\rangle$ and $B(y,x) = \langle T^*y, x\rangle$, so that we have

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

If taking conjugates , and can be see that T^* is the operator.

Remarks .[13]

If T is a linear operator with bounds then ,T $= 0$ iff , $\langle Tx, y\rangle = 0 \ \forall\, x, y \in H . T = 0$ means $Tx = 0$ for every $x \in H$ and thus $\langle Tx, y\rangle = \langle 0, y\rangle = 0$. Now if , $\langle Tx, y\rangle = 0$ for all $x, y \in H$ implies $Tx = 0 \ \forall\, x \in H$, which ,can be write T $= 0$.

Now showing some general properties of Hilbert adjoint operators.

Theorem (3.3.2)[19]

If H_1 , H_2 are Hilbert spaces , α any scalar and $S\colon H_1 \longrightarrow H_2$ and $T\colon H_1 \longrightarrow H_2$ are a bounded linear operators. Then

1) $\langle A^*y, z\rangle = \langle w, Az\rangle \qquad (z \in H_1 , w \in H_2)$
2) $(S + A)^* = S^* + A^*$
3) $(\alpha A)^* = \bar{\alpha} A^*$
4) $(A^*)^* = A$
5) $\|A^*A\| = \|AA^*\| = \|A\|^2$
6) $A^*A = 0 \; if and\ only\ if\ A = 0$
7) $(SA)^* = A^*S^* \qquad (assuming\ H_2 = H_1)$.

Proof

1) We have $\langle A^*w, z\rangle = \overline{\langle z, A^*w\rangle}\ then$

$$\overline{\langle z, A^*w\rangle} = \overline{\langle Az, w\rangle} = \langle w, Az\rangle.$$

2) For all z and w ,

$$\langle z, (S + A)^*w\rangle = \langle (S + A)z, w\rangle$$

$$= \langle Sz, w\rangle + \langle Az, w\rangle$$

$$= \langle z, S^* w\rangle + \langle z, A^* w\rangle$$

$$= \langle z, (S^* + A^*)w\rangle.$$

Hence $(S + A)^* w = (S^* + A^*)w$ for all w which is $(S + A)^* = (S^* + A^*)$

3) Now

$$\langle (\alpha A)^* w, z\rangle = \langle w, (\alpha A)x\rangle$$

$$= \langle w, \alpha(Az)\rangle$$

$$= \bar{\alpha}\langle w, Az\rangle$$

$$= \bar{\alpha}\langle A^* w, z\rangle$$

$$= \langle \bar{\alpha} A^* w, z\rangle.$$

And this hold for all $w \in H_2$ and obtained $(\alpha A)^* = \bar{\alpha} A^*$.

4) For all $z \in H_1$ and $z \in H_2$ we have

$$\langle (A^*)^* z, w\rangle = \langle z, A^* w\rangle = \langle Az, w\rangle$$

This implies that

$$\langle ((A^*)^* - A)z, w\rangle = 0 \quad \text{for all } w \in H_2,$$

and

$$(A^*)^* - A = 0.$$

Hence

$$(A^*)^* = A.$$

5) We see that $A^* A: H_1 \longrightarrow H_1$, but $AA^*: H_2 \longrightarrow H_2$

By the Schwarz inequality,

$$\|Az\|^2 = \langle A, Az\rangle = \langle A^* Az, z\rangle$$

$$\leq \|A^* Az\|\|z\| \leq \|A^* A\|\|z\|^2.$$

Taking the supremum over all z of norm1, hence

$$\|A^2\| \leq \|A^* A\|.$$

We thus have

$$\|A^2\| \leq \|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2.$$

Hence $\|A^*A\| = \|A\|^2$

Replacing A by A^* , we have

$$\|A^{**}A^*\| = \|A^*\|^2 = \|A\|^2.$$

Here $A^*A = A$ so that

$$\|A^*A\| = \|AA^*\| = \|A\|^2.$$

6) If $A^*A = 0$, then $\|A\|^2 = \|AA^*\| = 0$ this implies that $A = 0$, but if $A = 0$, then $\|AA\| = \|A\|^2 = 0$ this implies that $A^*A = 0$.

7) $\langle z, (SA)^*w\rangle = \langle (SA)z, w\rangle = \langle Az, S^*w\rangle = \langle z, A^*S^*w\rangle$.
Hence $(SA)^*w = A^*S^*w$ for all $w \in H_1 = H_2$.

Definition (3.3.2)

If A is algebra over $\mathbb{C}$. An **involution** is a mapping $T \longrightarrow T^*$of A into itself that holds, $\forall\, T, S \in A$ and every $\alpha \in \mathbb{C}$.

$$T^{**} = T\,, (T+S)^* = T^* + S^*, (\alpha T)^* = \bar{\alpha}T^*\,, (TS)^* = S^*T^*.$$

An algebra with an involution is called an **a* algebra space**. A $\boldsymbol{normed}^*$ **algebra** is a normed algebra with an involution.

A **C^*-algebra** is a Banch algebra A that has an involution satisfying $\|T^*T\| = \|T\|^2$.

$$\|T\|^2 = \|T^*T\| \leq \|\mathrm{T}^*\|\|\mathrm{T}\|\,,$$

which implies $\|\mathrm{T}\| \leq \|\mathrm{T}^*\|$ provided $T \neq 0$.

Replacing T by $T^*$and by using $T^{**} = T$, we obtain $\|\mathrm{T}^*\| \leq \|\mathrm{T}\|$ .Thus , $\|\mathrm{T}\| = \|\mathrm{T}^*\|$ for $T \in A$, since the equality is trivally true when T=0 .

Remak[15].

The true analogues of complex numbers are normed operators ; Note that

$$T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i},$$

where $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are self-adjoint and

$$T^* = \frac{T+T^*}{2} - i\frac{T-T^*}{2i}.$$

Real and imaginary parts of T are the operators $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$.

Next we give some examples.

Examples (3.3.1)

1) Let is $\mathbb{C}$ with conjugacy $\mathbb{C}^N$ has an involution

$$(Z_1, \dots, Z_N)^* = (\overline{Z_1}, \dots, \overline{Z_N})$$

This example extends to l^∞ .

2) $C[0,1]$ with conjugacy $\bar{f}(z) = \overline{f(z)}$.
3) Define T $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by setting$(Tx)_i = \sum_{j=1}^n \alpha_{ij} x_j$, if $H = \mathbb{C}^n$ the Hilbert space of finite dimension n, and $\{e_1, e_2, \dots, e_n\}$be the common orthonormal basis for H.

T is obviously linear and bounded as a result. Given that the inner product in$\mathbb{C}^n$ is $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$

$$\langle Tx, y \rangle = \sum_{i=1}^n (Tx)_i \overline{y_i}$$

$$= \sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} x_j \right) \overline{y_i}$$

$$= \sum_{j=1}^n x_j \overline{\sum_{i=1}^n \overline{\alpha_{ij}}\, y_i}$$

$$= \langle x, T^* y \rangle,$$

where $(T^* y)_j = \sum_{i=1}^n \overline{\alpha_{ij}}\, y_i$. The adjoint of T.

3.4 Special Classes of Operators.

"The classes of bounded linear operators of significant practical value have been investigated in this section using the Hilbert adjoint operator, which is defined as follows". [19].

Definition (3.4.1)

Let T a bounded linear operator on a Hilbert space H, $T: H \longrightarrow H$ is said to be

1) T is Hermitian or self – adjoint if $T^* = T$,
2) If T is bijective and $T^* = T^{-1}$, then T is unitary
3) Let $T^*T = TT^*$ then T be normal

Next we give some examples.

Examples (3.4.1)

1) Since self – adjoint and unit elements are normal.
2) Any $z \in \mathbb{C}$ is normal ; it is self – adjoint only when $z \in \mathbb{R}$ it is unitary when $|z| = 1$.
3) The operator T^*defined by $T^*x = \bar{\alpha}x$, $x \in H$,is the adjoint of the operator $T \in B(H)$ such that $Tx = \alpha x$, $x \in H$ and $\alpha \in \mathbb{C}$, Indeed , for $x, y \in H$, $\langle x, T^*y\rangle = \langle Tx, y\rangle = \langle \alpha x , y\rangle = \langle x, \bar{\alpha}y\rangle$.

Thus $\langle x, \langle T^* - \bar{\alpha}I\rangle y\rangle = 0$ consequently , $T^* = \bar{\alpha}I$.

4) Let S ,T are self – adjoint , then so are $S + T$, $\alpha T(\alpha \in \mathbb{R})$, $p(T)$ for any real polynomial p and T^{-1} if it exists but ST is self –adjoint iff ST=TS.

Theorem (3.4.1)[13]

If the operator $T: H \longrightarrow H$ is a bounded on H . Then

1) Let T be self – adjoint ,then $\langle Tx, x\rangle$ $\forall x \in H$ be real.
2) Let H be complex , $\langle Tx, x\rangle$ $\forall x \in H$ is real , then T is self – adjoint.

Proof

1) Let T be self – adjoint , hence,

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle \ \forall x.$$

Hence $\langle Tx, x\rangle$ is real since it equals its complex conjugate.

2) Let $\langle Tx, x\rangle$ be real for all x , then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle.$$

Hence

$$0 = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle (T - T^*)x, x\rangle$$

and $T - T^* = 0$ since H is complex. Then $T = T^*$.

Remark[15].

1) The previous proposition Part (2) is false if it only supposed that it is a real Hilbert space . The example ; if

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ on } \mathbb{R}^2,$$

then $\langle Tx, x\rangle = 0\ \forall x \in \mathbb{R}^2$.

However , $T^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T$.

2) Let $T \in B(H)$, then T^*T and $T + T^*$ are self –adjoint .

Theorem (3.4.2)[19]

When two bounded self-adjoint linear operators on a Hilbert space are combined to form S and T, H is only self-adjoint if and only if the operators commute, resulting in $ST = TS$.

Proof

If ST is self adjoint , then $(ST)^* = ST$ *but* $(ST)^* = T^*S^* = TS$.

Now if $ST = TS$,then

$$(ST)^* = T^*S^* = TS = ST.$$

This implies ST is self – adjoint.

Theorem (3.4.3)[13]

If (T_n) is a series of bounded self-adjoint linear operators $T_n: H \longrightarrow H$ on a Hilbert space H, then the limit operator T is a bounded self-adjoint linear operator on H if (T_n)converges, such that , $T_n \longrightarrow T$, so that, $\|T_n - T\| \longrightarrow 0$, where $\|\cdot\|$ is the norm on the space $B(H, H)$.

Proof

By follows $\|T - T^*\| = 0$.

$$\|T_n{}^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$$

and obtain by the theorem (2.2.1) part (2) in $B(H, H)$

$$\|T - T^*\| \leq \|T - T_n\| + \|T_n - T_n{}^*\| + \|T_n{}^* - T^*\|$$

$$= \|T - T_n\| + 0 + \|T_n - T\|$$

$$= 2\|T_n - T\| \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

Hence $\|T - T^*\| = 0$ and $T^* = T$.

Theorem (3.4.4)[15]

If the operators $U: H \longrightarrow H$ and $W: H \longrightarrow H$ by unitary . Then $U , W\ unitary \implies UW , U^{-1}$ unitary .

Unitary elements have unit norm , $\|U\| = 1$, provided $H \neq \{0\}$.

Proof

If U_n are unitary and $U_n \longrightarrow T$, then by continuity the involution , $U_n{}^* \longrightarrow T^*$ since $U_n{}^* U_n = 1 = U_n U_n{}^*$ become $T^*T = 1TT^*$ in the limit , that is $T^{-1} = T^*$ for any $U, W \in U(x)$, UW and $U^* = (U^{-1})$ are also unitary

$$(UW)^* = W^*U^* = W^{-1}U^{-1} = (UW)^{-1}$$

$$U^{**} = \mathrm{U} = (U^{-1})^{-1} = (U^*)^{-1}$$

Finally $$\|U\|^2 = \|U^*U\| = \|1\| = 1.$$

Lemma (3.4.5)[19]

If H is a complex Hilbert space , $T: H \longrightarrow H$ be linear operator on H and a bounded such that $\langle Tw, w\rangle\ \forall w \in H$, then T=0.

Proof

For $w, y \in H$,

$$\langle Tw, y\rangle = \frac{1}{4}\{\langle T(w+y), w+y\rangle - \langle T(w-y), w-y\rangle + i\langle T(w+iy), w+iy\rangle - i(T(w-iy), w-iy)\}.$$

Since $\langle Tw, w\rangle = 0$ for all $w, y \in H$, it follows that $\langle Tw, y\rangle = 0 \; \forall \; w, y \in H$ .setting $y = Tw,$

Thus $\|Tw\| = 0$ for every $w \in H,$ so $Tw = 0 \; \forall x \in H.$ consequently, T=0.

Definition (3.4.2)

T is positive semidefinite, let $T \in B(H)$be such that $T^* = T$ if for each $\in H$, $\langle Tx, x\rangle \geq 0$. If T is positive definite and $\langle Tx, x\rangle > 0$ for every nonzero $x \in H$. They are often referred to as strictly positive and positive operators.

Theorem (3.4.6)[7]

If $, T \in B(H)$, when a complex Hilbert space is H , if $ST = TS$ then their product ST is positive such that $S \geq 0$ $, T \geq 0$.

Proof

suppose $ST = TS$ and we show that $\langle STx, x\rangle \geq 0$ for all $x \in H$. Let S=0 , the inequality holds. If $\neq 0$. Set $S_1 = S/\|S\|$ $, S_2 = S_1 - {S_1}^2, \dots, S_{n+1} = S_n - {S_n}^2$, ..., for each S_i be self –adjoint . To prove , any $i = 1,2,$ ..., $0 \leq S_i \leq I$ .For $i = 1$ and $x \in H,$

$\langle S_1 x, x\rangle = \langle (S/\|S\|)x, x\rangle = \langle Sx, x\rangle/\|S\| \leq \|\mathrm{Sx}\|\|\mathrm{x}\|/\|S\| \leq \|\mathrm{x}\|^2 = \langle \mathrm{x}, \mathrm{x}\rangle$;

So , $\langle (I - S_1)x, x\rangle \geq 0$.

suppose that $0 \leq S_k \leq I$. Then $\langle {S_k}^2(I - S_k)x, x\rangle = \langle (I - S_k)S_k x, S_k x\rangle \geq 0$, that is ,

${S_k}^2(I - S_k) \geq 0$. similarly , it can be shown that $S_k(I - S_k)^2 \geq 0$. Consequently,

$S_{k+1} = {S_k}^2(I - S_k) + S_k(I - S_k)^2 \geq 0$ and $I - S_{k+1} = (I - S_k) + {S_k}^2 \geq 0$ by Thus ${S_k}^2 \geq 0$ where $S_k \geq 0$. This completes the argument when $0 \leq S_k \leq I$.

To observe that. Now consider the general case

$$S_1 = {S_1}^2 + S_2 = {S_1}^2 + {S_2}^2 + S_3 = \cdots = {S_1}^2 + {S_2}^2 + \cdots + {S_n}^2 + S_{n+1}.$$

Since $S_{n+1} \geq 0$, this implies

$${S_1}^2 + {S_2}^2 + \cdots + {S_n}^2 = S_1 - S_{n+1} \leq S_1 \ .$$

By the definition of $\leq$ and $S_i = {S_i}^*$, that is

$$\sum_{i=1}^{n} \|S_i x\|^2 = \sum_{i=1}^{n} \langle S_i x, S_i x\rangle = \sum_{i=1}^{n} \langle {S_i}^2 x, x\rangle$$

$$\leq \langle S_1 x, x\rangle.$$

Since n is arbitrary , the infinite series $\sum_{i=1}^{\infty} \|S_i x\|^2$ converges , which implies

$$\|S_i x\| \longrightarrow 0$$

and hence $S_i x \longrightarrow 0$.

Since

$$\langle \sum_{i=1}^{n} {S_i}^2 x, x\rangle = (S_1 - S_{n+1})x \longrightarrow S_1 x \qquad as\ n \longrightarrow \infty.$$

Since the sums and products of $S_1 = \|S\|^{-1}S$and S and T commute, S_i commutes with T.

$$\langle STx, x\rangle = \|S\| \langle S_1 Tx, x\rangle$$

$$= \|S\| \langle TS_1 x, x\rangle$$

$$= \|S\| \, \langle T \lim_n \sum_{i=1}^{n} {S_i}^2 \, x \, , x\rangle$$

$$= \|S\| \lim_n \sum_{i=1}^{n} \langle T{S_i}^2 x, x\rangle$$

$$= \|S\| \lim_n \sum_{i=1}^{n} \langle TS_i x, S_i x\rangle$$

$$\geq 0,$$

Using $S = \|S\|S_1$,and $T \geq 0$. Thus , $\langle STx, x\rangle \geq 0$ for all $x \in H$

Definition (3.4.3)

If linear operator $T_n: H \longrightarrow H$ is bounded on a Hilbert space H , $n = 1,2, ...$ and $\{T_n\}_{n\geq 1}$ is a sequence of bounded linear self – adjoint operators defined in a Hilbert space H,

the sequence $\{T_n\}_{n\geq 1}$ is called **increasing .**

[resp . decreasing] if $T_1 \leq T_2 \leq \cdots$ [resp . $T_1 \geq T_2 \geq \cdots$].

Theorem (3.4.7) [19]

Let $T \in B(H)$ and ≥ 0 . Then , there is a unique $V \in B(H)$ with $V \geq 0$ and $V^2 = T$.

Furthermore, every bounded operator that commutes with T also commutes with V.

Proof

Let $= 0$, then take $V = 0$.we suppose, $\|T\| \leq 1$. for any positive T and $z \in H,$

$$\langle Tz, z\rangle \leq \|Tz\|\|z\| \leq \|T\|\|z\|^2 = \|T\|\langle z, z\rangle\ ,$$

Which implies

$$\langle T/\|T\|\, z, z\rangle \leq \langle z, z\rangle\ , z \in H$$

and therefore , $T/\|T\| \leq I$. Hence, we may claim that there exists a positive operator V so that $V^2 = T/\|T\|$.

Conclusion that$\|T\|^{\frac{1}{2}}V$ is a positive square root of T .

And $I - T$ is self – adjoint ,

$$\|I - T\| = \sup_{\|z\|\neq 0} \frac{|\langle (I - T)z, z\rangle|}{\|z\|^2} = \sup_{\|z\|=1} |\langle (I - T)z, z\rangle| \leq 1.$$

Since the series

$$I + \alpha_1(I - T) + \alpha_2(I - T)^2 + \cdots \quad ,$$

converges in norm to an operator V We can be obtain that $V^2 = I - (I - T) = T$.

Furthermore ,since $0 \leq (I - T) \leq I$, we have

$$0 \leq \langle (I - T)^n z, z \rangle \leq 1,$$

for all $z \in H$ with $\|z\| = 1$. Thus ,

$$\langle Vz, z \rangle = 1 + \sum_{n=1}^{\infty} \alpha_n \langle (I - T)^n z, z \rangle$$

$$\geq 1 + \sum_{n=1}^{\infty} \alpha_n \text{ , using } \alpha_n < 0$$

$$= 0 \quad \text{, for all } n \geq 1$$

As the value of the series $1 + \sum_{n=1}^{\infty} \alpha_n S^n$ at $s = 1$,which is $1 + \sum_{n=1}^{\infty} \alpha_n$,is zero, the sum of the series is also zero.hence $, V \geq 0$.

We do not need the restriction that $\|T\| \leq 1$. If $S \in B(H)$ is such that $T = TS$.

Then , $S(I - T)^n = (I - T)^n S$ and consequently , $SV = VS$. To show that S is unique.

assume there is $\grave{V}$,with $\grave{V} \geq O$ and $\left(\grave{V}\right)^2 = T$. Then

$$\grave{V}T = \left(\grave{V}\right)^3 = T\grave{V} \ ,$$

T commutes with V, thus $\grave{V}$ commutes with T. Also , $\left(V - \grave{V}\right)V\left(V - \grave{V}\right) + \left(V - \grave{V}\right)\grave{V}\left(V - \grave{V}\right) = \left(V^2 - \grave{V}^2\right)\left(V - \grave{V}\right) = O.$

Due to the fact that both terms on the left are positive and equal to zero, their difference $\left(V - \grave{V}\right)^3 = O$. So $V - \grave{V}$ is hence self – adjoint ,

It hence

$$\left\|\left(V - \grave{V}\right)\right\|^2 = \left\|\left(V - \grave{V}\right)\left(V - \grave{V}\right)\right\| = \left\|\left(V - \grave{V}\right)^2\right\|.$$

And

$$\left\|\left(V - \grave{V}\right)\right\|^4 = \left\|\left(V - \grave{V}\right)^2\right\|^2 = \left\|\left(V - \grave{V}\right)^4\right\| \text{ , so } V - \grave{V} = O.$$

Example(3.4.2)

In $l^2[0,1]$, the multiplication operator

$$(Tx)(t) = tx(t) \quad , 0 < t < 1 \ , x \in l^2[0,1]$$

has the square root S , where

$$(Sx)(t) = \sqrt{tx}(t) \ , \ 0 < t < 1 \ , \ x \in l^2[0,1].$$

Theorem (3.4.8)[19]

If $T \in B(H)$ is self – adjoint and $\in \mathbb{N}$, then $\|T^n\| = \|T\|^n$.

Proof

Let $T = 0$. So may take $\|T\|^m > 0 \ \ \forall m \in \mathbb{N}$.

If $n = 1$ is trival.For $= 2$, we obtain

$$\|T^2\| = \|T^*T\| = \|T\|^2.$$

This says that, when k=1 ,the equality $\left\|T^{2^k}\right\| = \|T\|^{2^k}$ holds. suppose this for some $k \in \mathbb{N}$. Then ,

$$\|T^{2k+1}\| = \left\|\left(T^{2^k}\right)^2\right\| = \left\|\left(T^{2^k}\right)^*\left(T^{2^k}\right)\right\| = \left\|T^{2^k}\right\|^2 = \left(\|T\|^{2^k}\right)^2 = \|T\|^{2^{k+1}}.$$

If follows by induction that

$$\left\|T^{2^k}\right\| = \|T\|^{2^k} \qquad \text{for all } k \in \mathbb{N}.$$

Now consider an arbitrary $n \in \mathbb{N}$. Choose $k \in \mathbb{N}$such that $< 2^k$, and put $m = 2^k - n$.Then , $0 \leq \|T^m\| \leq \|T\|^m \neq 0$ and $0 \leq \|T^n\| \leq \|T\|^n$.

If it were to be the case that $\|T^n\| < \|T\|^n$,then it follow that

$$\left\|T^{2^k}\right\| = \|T^{n+m}\| \leq \|T^n\| \cdot \|T^m\| < \|T\|^n\|T\|^m = \|T\|^{n+m} = \|T\|^{2^k},$$

Contardicting what was proved earlier by induction. Thus, $\|T^n\| = \|T\|^n$.

Conclusion and Recommendation

It has been concluded that the transforming of linear independent sets into orthogonal sets, and transforming these sets into orthonormal sets in inner product spaces by using Gram – Schmidt process.

It can be determined the linear combination for the elements of orthonormal sequences by using Bessel inequality.

Riesz`s theorem shows representing bounded linear functional on Hilbert spaces by inner product .

The theorem (3.4.3) illustrates that the limit of sequence of bounded self – adjoint operators on such is self – adjoint bounded linear operator.

As the researcher has recommended on the necessity to continue searching in such topic in order to get the whole coverage of all sides of Hilbert space, like studying compacts and the spectrum of Hilbert spaces.

الملخص

في هذا البحث تمت دراسة بعض المفاهيم الأساسية المتعلقة بفضاءات الضرب الداخلي ومنها التعامد والتعامد الناظمي والجمع المباشر اللذان يلعبان دوراً كبيراً في بناء فضاءات الضرب الداخلي ، ثم بعد ذلك تم تعريف فضاءات هيلبرت وإعطاء أمثلة بسيطة عليها وفي ذلك تم التطرق إلى بعض المبرهنات الأساسية ذات العلاقة بفضاءات الضرب الداخلي وفضاءات هيلبرت مثل متباينة بيسل ومبرهنة جرام شميدت ، وكذلك تم تقديم خصائص المؤثرات الخطية والداليات الخطية والمؤثرات المرافقة و تأثيراتها على فضاءات هيلبيرت التي تلعب دورا هاما في التحليل الدالي ووصلنا إلى أن الداليات الخطية على فضاءات هيلبرت ماهي إلا الضرب الداخلي.

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