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Alasmarya Islamic University

Faculty of Science

Department of Mathematics

Some geometrical and analytical properties for certain classes of multivalent starlike functions

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By

Nouriyah Ali Nouh Abouthfeerah

Supervisor

Dr. Somia Muftah Amsheri

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List of symbols

A	Class of normalized analytic functions in the open unit disk $\mathcal{U}$
A_n	Class of analytic functions in $\mathcal{U}$ of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$
$A(p)$	Class of analytic and p-valent functions in $\mathcal{U}$
$\mathbb{C}$	Complex plane
$C(\alpha)$	Class of univalent convex functions of order α with negative coefficients
$C(p, \alpha)$	Class of p-valent convex functions of order α with negative coefficients
$C(\phi)$	Class of univalent convex functions with respect to the function ϕ
$C_b(\phi)$	Class of univalent convex functions of complex order *b* with respect to the function ϕ
$\mathcal{C}$	Class of univalent close-to-convex functions
$\mathcal{C}(\alpha)$	Class of univalent close-to-convex functions of order α
$\mathcal{C}(p)$	Class of p-valent close-to-convex functions
$\mathcal{C}(p, \alpha)$	Class of p-valent close-to-convex functions of order α
D	Domain
$D_{0,z}^{\lambda}$	Fractional derivative operator of order λ
D^m	Sălăgean differential operator
D_p^m	Generalized Sălăgean differential operator
D_δ^m	Al-Oboudi differential operator
$D_{\delta,p}^m$	Generalized Al-Oboudi differential operator
$\mathcal{H}(\mathcal{U})$	Class of analytic functions in $\mathcal{U}$

J_c	Bernardi-Libera-Livingston integral operator
$J_{c,p}$	Generalized Bernardi-Libera-Livingston integral operator
$J_{0,z}^{\lambda,\mu,\eta}$	Generalized fractional derivative operator
K	Class of univalent convex functions
$K(\alpha)$	Class of univalent convex functions of order α
K_b	Class of univalent convex functions of complex order b
$K(p)$	Class of p-valent convex functions
$K(p,\alpha)$	Class of p-valent convex functions of order α
$K_{b,p}$	Class of p-valent convex functions of complex order b
$\mathcal{K}(z)$	Koebe function
$\mathbb{N}$	Set of all positive integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$M_{0,z}^{\lambda,\mu,\eta,p}$	Generalized fractional derivative operator for p-valent functions
$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}$	Generalized differential operator for p-valent functions
$\mathcal{P}$	Class of functions with positive real part
$\mathcal{P}(\alpha)$	Subclass of the class $\mathcal{P}$ which satisfy Re $\{p(z)\} > \alpha$
$\mathcal{P}(\mathrm{A},\mathrm{B})$	Class of Janowski functions
$P(z)$	Möbius function
$P_{0,z}^{\lambda,\mu,\eta} f(z)$	Generalized fractional derivative operator for univalent functions
$\mathbb{R}$	Set of all real numbers
Re	Real part of a complex number
S	Class of normalized univalent functions
S^*	Class of univalent starlike functions
$S^*(\alpha)$	Class of univalent starlike functions of order α
S_b^*	Class of univalent starlike functions of complex order b

$S^*(p)$	Class of p-valent starlike functions
$S^*(p,\alpha)$	Class of p-valent starlike functions of order α
$S^*_{b,p}$	Class of p-valent starlike functions of complex order b
$S^*(\phi)$	Class of univalent starlike functions with respect to the function ϕ
$S^*_b(\phi)$	Class of univalent starlike functions of complex order b with respect to the function ϕ
$S^*_p(\phi)$	Class of p-valent starlike functions with respect to the function ϕ
$S^*_{b,p}(\phi)$	Class of p-valent starlike functions of complex order b with respect to the function ϕ
T	Class of univalent functions with negative coefficients
$T^*(\alpha)$	Class of univalent starlike functions of order α with negative coefficients
$T(p)$	Class of p-valent functions with negative coefficients
$T^*(p,\alpha)$	Class of p-valent starlike functions of order α with negative coefficients
$\mathcal{U}$	Open unit disk $\{z \in \mathbb{C},\ \lvert z\rvert < 1\}$
$\overline{\mathcal{U}}$	Closed unit disk $\{z \in \mathbb{C},\ \lvert z\rvert \le 1\}$
$\prec$	Subordinate to
$f * g$	Hadamard product (or convolution) of f and g
$(\lambda)_n$	Pochhammer symbol
$\Gamma(\lambda)$	Gama function
Ω	Class of all analytic functions of the form $$w(z) = w_1 z + w_2 z^2 + \cdots, (z \in \mathcal{U})$$
$\Omega^{\lambda}_{0,z} f(z)$	Generalized fractional derivative operator for univalent functions

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Abstract

The main objective of this research is to obtain several analytic and geometric properties of analytic and multivalent (p-valent) starlike functions defined in the open unit disk associated with certain linear operator by introducing certain classes and deriving some properties. In this thesis, a wide class of problems is investigated. First, Fekete-Szegö problems for functions belonging to some classes of p-valent starlike functions are solved. In addition, numerous starlikeness and convexity conditions of p-valent functions are obtained. Finally, certain classes of p-valent starlike functions with negative coefficients are defined, in obtaining, coefficient bounds, distortion properties, convolution properties, closure properties, extreme points, radius of close-to-convexity, radius of starlikeness, radius of convexity, class-preserving integral operators and integral means inequalities.

Chapter 1

Introduction

The thesis consists of five chapters. The purpose of this chapter is to give primitive background and motivations for the remaining chapters.

1.1 Review of literature

This section is devoted to give a brief history of review of literature which deals with the conceptual framework of the present research problem.

The studies reviewed focus on how interest to introduce new classes of analytic univalent and multivalent (or p-valent) functions and investigate their properties. Also, what effect of linear operators on functions belonging to those classes. The review of related literature studied by the researcher is divided into the following categories:

- Univalent functions
- Multivalent functions
- Coefficients bounds problem
- Starlikeness and convexity conditions
- Linear operators

1.1.1 Univalent functions

Complex analysis is one of the main branches of mathematics, its roots back to the beginning of the 19th century [15], [18], [19] and scientists have taken great interest in it since the discovery of the space of complex numbers, because it has applications in branches of mathematics and other science. One may refer to the works papers presented by the most famous mathematicians such as Euler, Gauss, Riemann, Cauchy, Weierstrass and others.

The geometric function theory is an old area of complex analysis that deals with geometric and analytic properties of analytic functions. It was set as a separately branch of complex analysis in the twenty century. Also, it has several applications in other areas such as modern mathematical physics, nonlinear integrable systems theory and the theory of partial differential equations.

The theory of univalent function in the open unit disk has been extensively studied in the mathematical literature [15], [18] since the beginning of the last century and is a classical problem of complex analysis which belongs to one of the most beautiful subjects in geometric function theory when appeared the first important paper by Koebe in 1907 on Riemann mapping theory and conformal mappings, to Gronwall's proof of the area Theorem in 1914, and to Bieberbach's conjecture for the coefficient problem in 1916 which was solved by Branges in 1985. By then, univalent function theory was a subject in its own right.

Moreover, to study the properties of a function on a simply connected domain D, it is so vast and complicated, the most obvious is to replace the arbitrary domain D by one that is convenient, and the most attractive selection is the open unit disk as a domain of definition of univalent function which has the advantage of simplifying the computations and leading to short and elegant formulas [15], [18], [19]. Various other terms are used for this concept (univalent), e.g. simple, or schlicht (the German word for simple).

The geometry theory of functions is mostly concerned with the study of properties of normalized univalent functions f which belong to the class S and defined in the open unit disk $\mathcal{U}$. The image domain of $\mathcal{U}$ under univalent function is of interest if it has some nice geometry properties. For example, a convex domain is an outstanding example of a domain with nice properties. Another example such domain is starlike with respect to a point. Certain subclasses of those analytic univalent functions which map $\mathcal{U}$ onto these geometric domains are introduced and their properties are widely investigated, for example, the classes K and S^* of univalent convex and univalent starlike functions, respectively. It was observed that both of these classes are related with each other through classical Alexander type relation [1] which says $f \in K$ if and only if $zf' \in S^*$. The special subclasses of these classes are the classes $K(\alpha)$ and $S^*(\alpha)$ of univalent convex and univalent starlike functions of order $\alpha, 0 \leq \alpha < 1$, respectively. If $\alpha = 0$, the classes of univalent convex and univalent starlike functions, respectively are obtained. These classes were first introduced by Robertson [51] and were studied subsequently by Schild [58], Pinchuk [45], Jack [23] and others.

Also, the classes of univalent convex and univalent starlike functions are closely related with the class $\mathcal{P}$ of analytic functions with positive real part, many problems are solved by using the properties of this class [21], [32], [49] and others.

1.1.2 Multivalent functions

The natural generalization of univalent function is p-valent (or multivalent) function which belongs to the class $A(p), p \in \mathbb{N}$ and defined in the open unit disk $\mathcal{U}$. If f is p-valent function with $p = 1$, then f it is univalent function. Also, the classes K and S^* of univalent convex and univalent starlike functions were extended to the classes $K(p)$ and $S^*(p)$ of p-valent convex and p-valent starlike functions, respectively by Goodman [17]. The special subclasses of these classes $K(p)$ and $S^*(p)$ are the classes $K(p, \alpha)$ and $S^*(p, \alpha)$ of p-valent convex and p-valent starlike functions of order $\alpha, 0 \leq \alpha < p$. If $\alpha = 0$, the classes of p-valent convex and p-valent starlike functions, respectively are obtained. The class $K(p, \alpha)$ was introduced by Owa [40] and the class $S^*(p, \alpha)$ was introduced by Patil and Thakare [44].

Notice that the studies reviewed focus on how interest to introduce new classes of analytic univalent and p-valent functions and investigate their properties such as coefficient bounds, distortion properties, extreme points, radii of starlikeness and convexity and others.

1.1.3 Coefficients bounds problem

In most cases, solutions of differential equations are usually expressed in series expansion form and it is desired that the series converge. Convergent of series depend largely on the coefficient of the expansions thus, it is of interest to research into coefficients bounds.

The famous coefficient problem is the Bieberbach's conjecture, it states that if $f \in S$, then $|a_n| \leq n$ for each $n \geq 2$. This conjecture was unsolved for about 70 years, although it had been proved in several special cases $n = 2,3,4,5,6$ and many other subclasses of S. In 1916, the first result was given by Bieberbach for $n = 2$ which satisfies $|a_2| \leq 2$. But finally, Louis de Brages settled it in 1985 [15], [18]. This result was used as an application to show that if f is univalent and normalized in the open unit disk and the image domain of $\mathcal{U}$ under f must cover the open disk with center at the origin and radius $\frac{1}{4}$. Because of several other applications on coefficient bounds, many authors have researched coefficient bounds for different subclasses of univalent and multivalent functions.

The problem of coefficient bounds is one of interesting problems which was studied by researchers for various classes of starlike and convex (p-valent starlike and p-valent convex) functions with negative coefficients in the open unit disk. Closely related to this problem is to determine how large the modulus of a univalent or p-valent function together with its derivatives can be in a particular class. Such results, referred to as distortion inequalities which provide important information about the geometry of functions in that class. The result which is as inequality is called sharp (best possible or exact) in sense, that it is impossible to improve the inequality (decrease an upper bound, or increase a lower bound) under the conditions given and it can be seen by considering a function such that equality holds. This function is called extremal function. A function belong to the class of functions is called an extreme point if it cannot be written as a proper convex combination of two other members of this class. The radius of starlikeness or convexity problem for a certain class of functions is to determine the largest disk $|z| < r$, i.e. the largest number of r , $0 < r \leq 1$ such that each function f in the class is starlike or convex in $|z| < r$. One may refer to the books by [15], [18], [19] and [37].

Those problems have attracted many mathematicians involved in geometric function theory, for example, Silverman [60] introduced and studied the classes $T^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions with negative coefficients of order α, $0 \leq \alpha < 1$, respectively. These classes were generalized to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ of p-valent starlike and convex functions with negative coefficients of order $\alpha, 0 \leq \alpha < p$ respectively, which were introduced by Owa [40], in order to derive the similar properties above. There are many contributions on connections between various subclasses of analytic univalent and multivalent functions were studied by researchers [10], [12], [42], [56], [59], [61], [63] and others.

Moreover, further studies on the generalized families of coefficient bounds. In 1933, Fekete and Szegö obtained the sharp bound of $|a_3 - \eta a_2^2|$ as a function of real parameter η for the class S of univalent functions [13]. The result is sharp in the sense that for each η there is a function in the class under consideration for which the equality holds. This is known as Fekete-Szegö inequality or Fekete-Szegö problem.

There are several results for this type in literature, each of them dealing with $|a_3 - \eta a_2^2|$ for various classes of functions. For example, Ma and Minda

[32] solved the Fekete-Szegö problem for univalent starlike and convex functions in the classes $S^*(\phi)$ and $C(\phi)$, respectively, for the univalent starlike function $\phi(z)$ with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0) = 1$ and $\phi'(0) > 0$. Also, Ravichandran et. al. [49] solved the Fekete-Szegö problem for univalent starlike functions of complex order in the class $S_b^*(\phi)$. The sharp bounds for the functional $\left|a_{p+2} - \eta a_{p+1}^2\right|$ have been solved for generalized class $S_{b,p}^*(\phi)$ of p-valent starlike functions by Ali et. al. [2]. These results were generalized [8] by making use of the fractional derivative operator. There are many papers on this problem that can refer to [26], [48] and others.

1.1.4 Starlikeness and convexity conditions

The problem of sufficient conditions for starlikeness and convexity is concerning to find conditions under which functions in certain class are starlike or convex, respectively. There are many works on the sufficient conditions for starlikeness and convexity of analytic functions, for example, Ruscheweyh and Sheil-Small [54] obtained many sufficient conditions for starlikeness and convexity. Also, Silverman [60] obtained the sufficient conditions for functions to be in the class $S^*(\alpha)$ or the class $K(\alpha)$. Further, Owa and Shen [41] introduced various sufficient conditions for starlikeness and convexity of univalent functions involving the fractional derivative operator $\Omega_{0,z}^{\lambda}$ by using results of Silverman [60] and by using results involving the Hadamard product (or convolution) due to Ruscheweyh and Sheil-Small [54]. These results were generalized by Raina and Nahar [47] to obtain useful results that deal with the starlikeness and convexity for the fractional derivative operator $P_{0,z}^{\lambda,\mu,\eta}$ of analytic and univalent functions.

Furthermore, there are other results for starlikeness and convexity conditions have been obtained by Jack's Lemma and Nonokawa's Lemma, for example, Irmak and Piejko [21] investigated some starlikeness and convexity of certain normalized functions which are analytic and univalent in the open unit disk.

Moreover, various results were extended by solving this problem for p-valent functions, for example, Owa [40] proved the sufficient conditions for analytic and p-valent functions to be in the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ for $p \in \mathbb{N}$ and $0 \leq \alpha < p$. Also, Amsheri and Zharkova [7] obtained many sufficient conditions for starlikeness and convexity of p-valent functions associated with the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p}$ by using known results for the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ due to Owa [40] and by using results involving the Hadamard product due to Ruscheweyh and Sheil-Small [54]. There are many other researchers [20], [22], [43] and others who studied starlikeness and convexity conditions.

1.1.5 Linear operators

The concept of the differentiation operator $D = d/dx$ is familiar to all who studied the elementary calculus, and for suitable function f, the *n*th derivative of f, namely $D^n f(x) = d^n f(x)/dx^n$ is well defined provided that n is a positive integer. In 1695, L'Hôpital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^v, where v could be rational or irrational, positive or negative, real or complex [33]. Thus the name fractional calculus became somewhat of a misnomer.

The fractional derivative operator has found interesting applications in the theory of analytic functions as well as other linear operators such as Sălăgean operator and Al-Oboudi operator have been applied in introducing various classes of analytic functions and obtaining several properties. For numerous works on this subject, one may refer to the works which were studied [3], [7], [8], [11], [16], [22], [27], [31], [39], [41], [47], [48], [49], [55], [57], [59], [63], [64] and others.

1.2 Research problem

The thesis is organized with solutions to a number of problems. For example, the following problems are considered:

1. Find the bounds for the functional $\left|a_{p+2} - \theta a_{p+1}^2\right|$ for p-valent functions belonging to certain classes?
2. What are the sufficient conditions for starlikeness and convexity of p-valent functions?
3. How does the series of coefficients influence the geometric and analytic properties of functions which belong to certain classes of p-valent starlike functions?

1.3 Research objectives

The main objective of this research is to obtain some geometrical and analytical properties for certain classes of p-valent starlike functions defined in the open unit disk by using differential operators. That are

1. To define some classes of p-valent functions and solve Fekete-Szegö problem for functions belonging to those classes.

2. To find sufficient conditions for p-valent functions to be starlike and convex.
3. To identify some classes of p-valent functions with negative coefficients and find coefficient bounds, distortion properties, convolution properties, closure properties, extreme points, radius of close-to-convexity, radius of starlikeness, radius of convexity, class-preserving integral operators and integral means inequalities.

1.4 Research methodology

The following analytical methods are proposed to be used to undertake the research work:

- Subordination between analytic functions.
- Methods arising from the convolution theory.

These methods are also proposed to be used to study theorems and to discuss the geometric properties of the defined classes.

1.5 Research motivations and outlines

In geometric function theory, the attention to geometrical and analytical properties for univalent and p-valent functions has been the main interest among authors. Hence there are many new subclasses and new properties of univalent and p-valent functions have been introduced. The study of operators plays a vital role in mathematics. To apply linear operators for univalent and p-valent functions and then study their properties, is one of the hot areas of current ongoing research in geometric function theory.

In this thesis, motivated by wide applications of linear operators in the study of univalent and p-valent functions [3], [7], [8], [11], [16], [22], [27], [31], [33], [39], [41], [47], [48], [49], [55], [57], [59], [63], [64] and others, we present a study regarding various properties of some classes of p-valent starlike functions such as sharp coefficient bounds, sufficient conditions for starlikeness and convexity and properties of classes of functions with negative coefficients.

By studying the theory of univalent and multivalent function and motivated by the linear operators, chapter 2 deals with the elementary concepts of univalent functions, multivalent functions and functions with a positive real part. This chapter also presents some classes of starlike, convex and close-to-convex (p-valent starlike, p-valent convex and p-valent close-to-convex) functions in the open unit disk and contains some definitions of linear operators.

Many authors have solved the classical result of Fekete and Szegö $|a_3 - \eta a_2^2|$ for various classes of analytic functions, motivated by the works [2], [8], [13], [26], [32], [48], [49] and others, we form chapter 3, which deals with Fekete-Szegö inequalities for well-known classes of p-valent functions. Furthermore, a new class of p-valent functions associated with generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}$ is introduced and Fekete-Szegö inequalities are obtained.

In addition, as a motivation of some works on starlikeness and convexity conditions due to [7], [20], [21], [22], [41], [43], [47], [54] and others, we form chapter 4, which leads to give conditions for p-valent functions and find some new conditions for p-valent functions associated with the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}$.

Several classes of univalent functions have been extended to the case of p-valent functions in obtaining some properties such as coefficient bounds, distortion properties, convolution properties, closure properties, extreme points, radius of close-to-convexity, radius of starlikeness, radius of convexity, class-preserving integral operators and integral means inequalities, motivated by the studies [10], [12], [40], [42], [56], [59], [60], [61], [63] and others, we form chapter 5, which deals with obtaining the properties of the well-known class of p-valent functions $T^*(p,\alpha)$ and a new class of p-valent functions defined by the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}$.

Chapter 2

The elementary concepts of analytic univalent and multivalent functions

The purpose of this elementary chapter is to review some of the general principles of Complex Analysis, which underlie the Geometric Function Theory of complex variable.

2.1 Univalent functions

In this section, some definitions and basic results concerning analytic and univalent functions in the open unit disk are presented.

Definition 2.1.1 [15], [19], [66]

1. A domain D is an open connected set in the complex plane $\mathbb{C}$. A domain is said to be simply connected if its complement is connected. Geometrically, a simply connected domain is not contained any holes.
2. A neighborhood of a set $D \subset \mathbb{C}$ is an open subset which contains D in the complex plane $\mathbb{C}$.
3. A function is a rule of correspondence between two sets such that there is a unique element in the second set assigned to each element in the first set.
4. A complex-valued function $f(z)$ of a complex variable is differentiable at a point $z_0 \in \mathbb{C}$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

5. A function $f(z)$ is analytic at z_0 if it is differentiable at z_0 and every point in some neighborhood of z_0. A function $f(z)$ is analytic on a

domain D if it is analytic at all points in D. The function $f(z)$ must have derivatives of all orders at z_0, and that $f(z)$ has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

which converges in some open disk centered at z_0 .

6. A function $f(z)$ is entire function if it is analytic in the whole complex plane $\mathbb{C}$.
7. The open unit disk $\mathcal{U}$ is the set of all points $z \in \mathbb{C}$ of modulus $|z| < 1$

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

The closed unit disk $\overline{\mathcal{U}}$ is the set of all points $z \in \mathbb{C}$ of modulus $|z| \leq 1$

$$\overline{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$$

8. The class $\mathcal{H}(\mathcal{U})$ is the set of all analytic functions in $\mathcal{U}$.
9. A function $f(z)$ which is an analytic on a domain D is said to be univalent there if it does not take the same value twice, that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D. In other words $f(z)$ is one-to-one mapping on D onto another domain.
10. A function $f(z)$ which is an analytic on a domain D is said to be locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0. It is evident that $f(z)$ is locally univalent at z_0 provided $f'(z_0) \neq 0$.

Definition 2.1.2 [12]

1. The class A_n, $n \in \mathbb{N} = \{1,2,3, ...\}$ is the subset of $\mathcal{H}(\mathcal{U})$ consisting of all functions of the form

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots, \quad (z \in \mathcal{U}) \tag{2.1.1}$$

for $n = 1$, the class $A_1 = A$ is the subclass of the class $\mathcal{H}(\mathcal{U})$ of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.1.2)$$

which are normalized by $f(0) = 0$ and $f'(0) = 1$.

2. The class S is the subclass of the class A consisting of univalent functions.
3. The class T is the subclass of the class S which consists of functions having non-zero coefficients, from the second on, are negative of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0; z \in \mathcal{U}) \qquad (2.1.3)$$

Definition 2.1.3 [65]

The functions $f_\alpha(z)$ are called rotations of the function $f(z)$ which belongs to the class S if for any real number α, the functions $f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z)$ are also in the class S.

Definition 2.1.4 [41], [43]

1. The Hadamard product (or convolution) of $f(z) \in A$ given by (2.1.2) and $g(z) \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.1.4)$$

is denoted by $(f * g)(z)$ and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.1.5)$$

2. The Hadamard product (or convolution) of $f(z) \in T$ given by (2.1.3) and $g(z) \in T$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \qquad (b_n \geq 0; z \in \mathcal{U}) \qquad (2.1.6)$$

is denoted by $(f * g)(z)$ and defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.1.7)$$

Example 2.1.5 [15]

The identity function $f(z) = z$ is in the class S.

Example 2.1.6 [15]

The function

$$f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \qquad (z \in \mathcal{U})$$

is in the class S. It maps $\mathcal{U}$ onto Re $z > -\frac{1}{2}$.

Example 2.1.7 [18]

The Koebe function $\mathcal{K}(z)$ is in the class S and given by

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = z + \sum_{n=2}^{\infty} nz^n, \quad (z \in \mathcal{U})$$

it maps $\mathcal{U}$ one-to-one onto the domain D that consists of the whole complex plane except for a slit along the half-line $(-\infty, -\frac{1}{4}]$.

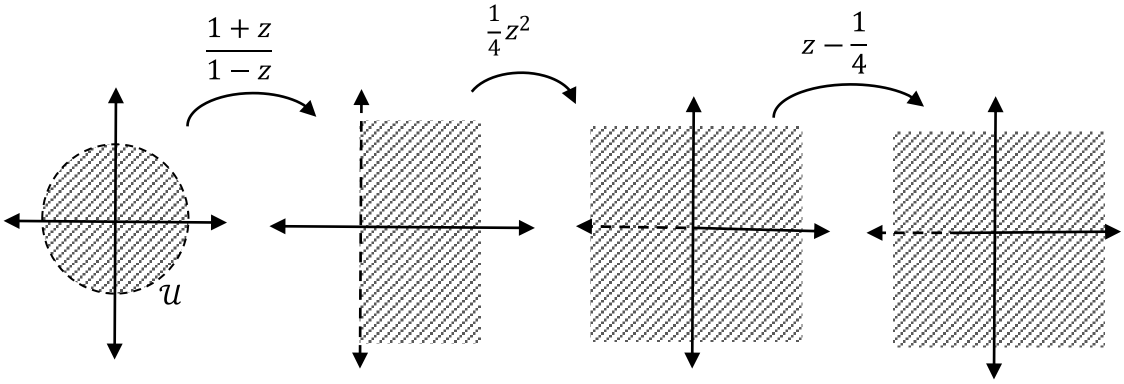


Figure 2.1: The Koebe function

The Koebe function and its rotations, play a very important role in the study of the class S. They often are the extremal functions for various problems in S.

In1916, Bieberbach proved the following theorem [18] about the estimating the second coefficient a_2 of a function of class S.

Theorem 2.1.8 (Bieberbach's Theorem)

If $f(z) \in S$, then $|a_2| \leq 2$. Equality holds if and only if $f(z)$ is a rotation of the Koebe function $\mathcal{K}(z)$.

The problem known as the "Bieberbach's conjecture" has played a central role in the development of the subject of univalent functions. Many interesting techniques in geometric function theory were developed to obtain various partial results on the Bieberbach's conjecture. The full Bieberbach conjecture was finally proved in 1985 by De Branges [14].

Theorem 2.1.9 (Bieberbach's Conjecture)

If $f(z) \in S$, then $|a_n| \leq n$ for all $n \geq 2$. If for any n, $|a_n| = n$, then $f(z)$ is a rotation of the Keobe function $\mathcal{K}(z)$.

The following theorem [46] is another coefficient problem which deals with the bounds of $|a_3 - a_2^2|$.

Theorem 2.1.10

If $f(z) \in S$, then $|a_3 - a_2^2| \leq 1$.

2.2 Multivalent functions

This section is devoted to present some concepts concerning the generalized univalent functions in the open unit disk, namely the multivalent (or $\boldsymbol{p}$-valent) functions.

Definition 2.2.1 [18]

A function $f(z)$ analytic in the open unit disk $\mathcal{U}$ is said to p-valent in $\mathcal{U}$, or multivalent of order $p, (p = 1, 2, ...)$ in $\mathcal{U}$ if the equation $w = f(z)$ has never more than p-solutions in $\mathcal{U}$ and there exists some w for which this equation has exactly p solutions. For $p=1$, the p-valent function $f(z)$ reduces to univalent function.

Definition 2.2.2 [40]

1. The class $A(p)$ is the set of all functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.2.1)$$

which are analytic and p-valent in the open unit disk $\mathcal{U}$.

2. The class $T(p)$ is a subclass of the class $A(p)$ consisting of all p-valent functions with negative coefficients of the form

$$f(z)=z^p-\sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (a_{p+n} \geq 0; p \in \mathbb{N}; z \in \mathcal{U}) \quad (2.2.2)$$

Definition 2.2.3 [7], [40]

1. The Hadamard product of $f(z) \in A(p)$ given by (2.2.1) and $g(z) \in A(p)$ given by

$$g(z)=z^p+\sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.2.3)$$

is denoted by $(f*g)(z)$ and defined by

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} b_{p+n}\, z^{p+n}, \quad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.2.4)$$

2. The Hadamard product of $f(z) \in T(p)$ given by (2.2.2) and $g(z) \in T(p)$ given by

$$g(z)=z^p-\sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}, \quad \left(b_{p+n} \geq 0; p \in \mathbb{N}; z \in \mathcal{U}\right) \quad (2.2.5)$$

is denoted by $(f*g)(z)$ and defined by

$$(f*g)(z)=z^p-\sum_{n=1}^{\infty} a_{p+n} b_{p+n}\, z^{p+n}, \quad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.2.6)$$

Example 2.2.4 [65]

The p-valent function $f(z) = z^p, \ p \in \mathbb{N}$ maps $\mathcal{U}$ onto $\mathcal{U}$, but each image point (except $w = 0$) has p different preimages. More picturesquely, $f(z) = z^p$ can be viewed as mapping $\mathcal{U}$ in the z-plane univalently onto a spiral-like surface with p layers (sheets) hovering over $\mathcal{U}$ in the w-plane.

2.3 Functions with positive real part

In this section, the concept of subordination between analytic functions in the complex plane is presented, which was developed by Littlewood [28], [29] and Rogosinski [52], [53]. Also, the class of all analytic functions with positive real part is defined. These functions map the open unit disk $\mathcal{U}$ onto right half-plane.

The following classical result, which popularly known as Schwarz's Lemma in the literature have been used in defining the subordination principle.

Schwarz's Lemma 2.3.1 [37]

Let the function $w(z)$ be analytic in $\mathcal{U}$ and let $w(0) = 0$. If $|w(z)| \leq 1$ ($z \in \mathcal{U}$) then $|w(z)| \leq |z|$. The equality can hold only if $w(z) \equiv kz$ and $|k| = 1$.

Definition 2.3.2 [19]

The class Ω is the set of all analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots, \qquad (z \in \mathcal{U})$$

such that $w(0) = 0$ and $|w(z)| < 1$. In other words, Ω consists precisely of those analytic functions on $\mathcal{U}$ which satisfy the hypotheses of the Schwarz's Lemma.

Definition 2.3.3 (Subordination principle) [18]

Let the functions $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. The function f is said to be subordinate to the function g (written $f \prec g$ or $f(z) \prec g(z)$), if there exists a Schwarz function $w \in \Omega$ such that $f(z) = g\big(w(z)\big), \ z \in \mathcal{U}$. Furthermore, if the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

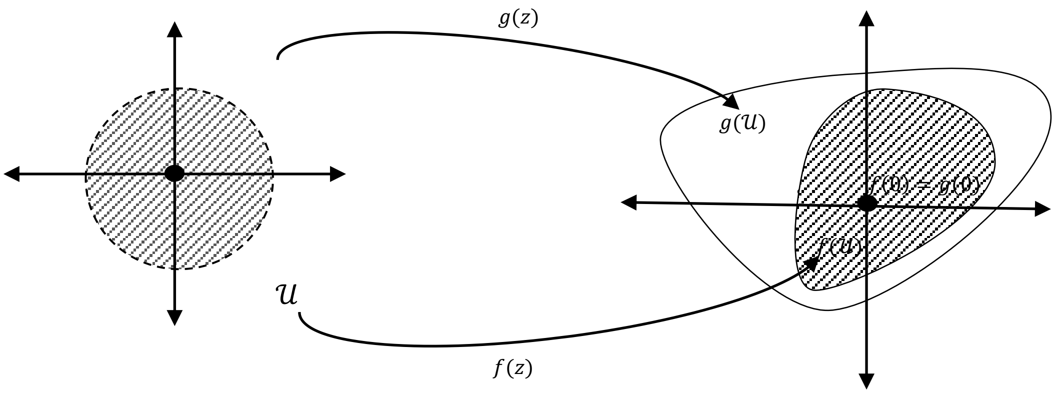


Figure 2.2: The function f is subordinate to the function g

Definition 2.3.4 [18]

The class $\mathcal{P}$ is the set of all analytic functions in $\mathcal{U}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.3.1)$$

which satisfy the conditions $p(0) = 1$ and Re $\{p(z)\} > 0$ $(z \in \mathcal{U})$. This class is usually called the Carathéodory class.

The following example shows that the class $\mathcal{P}$ need not to be univalent.

Example 2.3.5 [46]

The function

$$p(z) = 1 + z^n \in \mathcal{P}, \qquad (z \in \mathcal{U})$$

for any $n \in \mathbb{N}$, but if $n \geq 2$, this function is not univalent.

Example 2.3.6 [18]

The Möbius function

$$P(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n, \qquad (z \in \mathcal{U})$$

this function is in the class $\mathcal{P}$, it is analytic and univalent in $\mathcal{U}$, and it maps $\mathcal{U}$ onto the right half-plane.

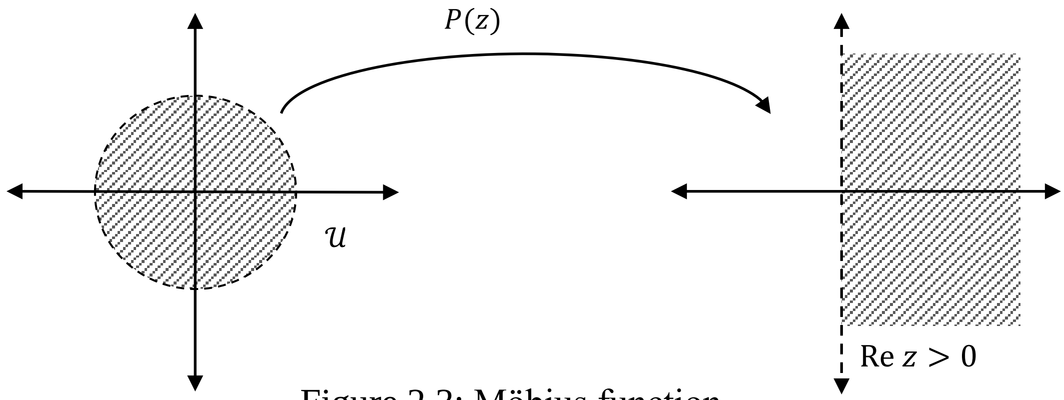


Figure 2.3: Möbius function

Definition 2.3.7 [18]

Any function $p(z)$ in the class $\mathcal{P}$ is called a function with positive real part in $\mathcal{U}$ and satisfies $p(z) \in \mathcal{P}$ if and only if $p(z) \prec \frac{1+z}{1-z}$.

Some special subclasses of the class $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied some became the well-known.

Definition 2.3.8 [18]

The class $\mathcal{P}(\alpha)$ is the subclass of the class $\mathcal{P}$ of analytic functions $p(z)$ for which satisfy $\mathrm{Re}\,\{p(z)\} > \alpha, 0 \le \alpha < 1$ and $z \in \mathcal{U}$. A functions $p(z) \in \mathcal{P}(\alpha)$ can be written as

$$p(z) = (1-\alpha)p_1(z) + \alpha, \qquad (p_1(z) \in \mathcal{P})$$

Definition 2.3.9

The class $\mathcal{P}(\mathrm{A},\mathrm{B})$ is the subclass of the class $\mathcal{P}$ of analytic functions $p(z)$ in $\mathcal{U}$ with $p(0) = 1$ for given arbitrary numbers A, B $(-1 \le \mathrm{B} < \mathrm{A} \le 1)$, and satisfy the following condition

$$p(z) \in \mathcal{P}(\mathrm{A},\mathrm{B}) \Leftrightarrow p(z) \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z}$$

The class $\mathcal{P}(\mathrm{A}, \mathrm{B})$ was introduced by Janowski [24]. In particular, for special selections of A and B, we have

1. $\mathcal{P}(1, -1) = \mathcal{P}$
2. $\mathcal{P}(1 - 2\alpha,\ -1) = \mathcal{P}(\alpha), \qquad (0 \leq \alpha < 1),$
3. $\mathcal{P}(1, 0)$ is the class of functions p(z) defined by $|p(z) - 1| < 1$.

2.4 Starlike and convex functions

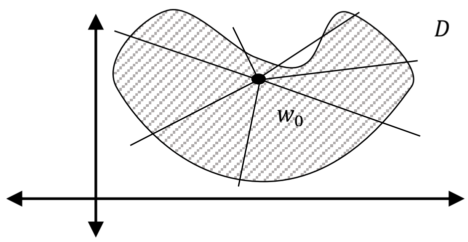
This section is devoted to study the most important subclasses of S, namely the classes of starlike and convex functions, which are closely related to the class $\mathcal{P}$. Both classes are defined by geometrical considerations, but both have very useful analytic characterizations.

Definition 2.4.1 [18]

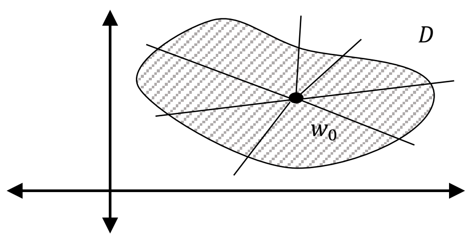
A domain D in $\mathbb{C}$ is said to be starlike with respect to a fixed point $w_o \in D$ if the closed line segment joining any point $w \in D$ to w_o lies entirely in D.

A function $f(z) \in S$ in $\mathcal{U}$ is said to be starlike with respect to w_o if $\mathcal{U}$ is mapped onto a starlike domain with respect to w_o. In the special case, when $w_o = 0$, the function $f(z)$ is said to be starlike with respect to the origin (or starlike).

The class of all functions of S which are starlike in the open unit disk $\mathcal{U}$ is denoted by S^*.



Domain is not starlike
with respect to w_o



Domain is starlike
with respect to w_o

Figure 2.4: Domain is starlike and domain is not starlike

The well-known analytical characterization for starlikeness is given as follows.

Theorem 2.4.2 [15]

If $f(z) \in S$. Then $f(z) \in S^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$, that is

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (z \in \mathcal{U}) \qquad (2.4.1)$$

Definition 2.4.3 [18]

A domain D in $\mathbb{C}$ is said to be convex if for all $w_1, w_2 \in D$, the closed line segment between w_1 and w_2 lies entirely in D. In other words, a domain D is said to be convex if it is starlike with respect to each of its points.

A function $f(z) \in S$ is said to be convex if $\mathcal{U}$ is mapped onto a convex domain.

The class of all functions of S which are convex in the open unit disk $\mathcal{U}$ is denoted by K.

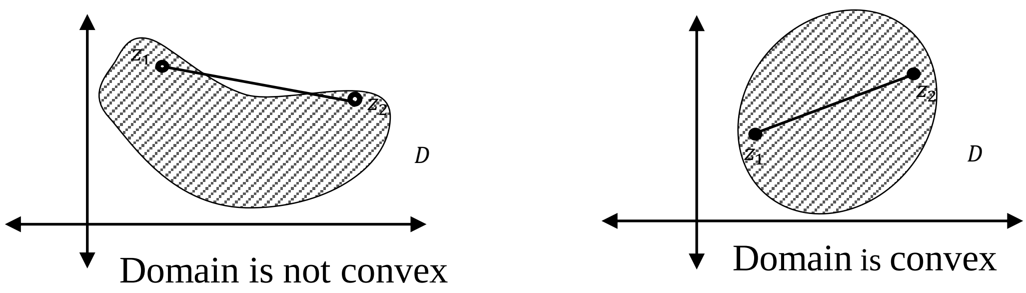


Figure 2.5: Domain is convex and domain is not convex

The following well-known analytical characterization for convexity is presented.

Theorem 2.4.4 [15]

If $f(z) \in S$. Then $f(z) \in K$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$, that is

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathcal{U}) \qquad (2.4.2)$$

The results given in Theorem 2.4.2 and Theorem 2.4.4 lead to a very useful and beautiful relationship between the classes S^* and K. This connection was first discovered in 1915 by Alexander [1] and is known as Alexander's Theorem.

Theorem 2.4.5

If $f(z) \in S$, then $f(z) \in K$ if and only if $zf'(z) \in S^*$.

Remark 2.4.1

Every convex function is starlike. Then $K \subset S^* \subset S$.

Example 2.4.6 [18]

The Koebe function $\mathcal{K}(z)$ is starlike but not convex.

Example 2.4.7 [18]

The function

$$f(z) = \frac{z}{1-z}$$

is convex.

The following result gives the relationship between the classes S and $\mathcal{P}$ which is known as Noshiro – Warschawski Theorem.

Theorem 2.4.8 [15]

If $f(z)$ is analytic function in a convex domain D and Re $\left(f'(z)\right) \geq 0$, then $f(z)$ is univalent on D.

In 1936, Robertson [51] introduced the following two subclasses of starlike and convex functions in the open unit disk, respectively, that are $S^*(\alpha)$ and $K(\alpha)$.

Definition 2.4.9

A function $f(z) \in S$ is said to be starlike function of order α, $0 \leq \alpha < 1$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}) \qquad (2.4.3)$$

The class of all starlike functions of order α, $0 \leq \alpha < 1$ is denoted by $S^*(\alpha)$. Notice that,

$$S^*(0) = S^*$$

Definition 2.4.10

A function $f(z) \in S$ is said to be convex function of order α, $0 \leq \alpha < 1$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}) \qquad (2.4.4)$$

The class of all convex functions of order α, $0 \leq \alpha < 1$ is denoted by $K(\alpha)$. Notice that,

$$K(0) = K$$

There is an Alexander type result [60] relating $S^*(\alpha)$ and $K(\alpha)$ which says the function $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$ for $0 \leq \alpha < 1$ and $z \in \mathcal{U}$.

In 1985, Nasr and Aouf [36] introduced the following subclass of starlike functions in the open unit disk, that is S_b^*.

Definition 2.4.11

A function $f(z) \in S$ is said to be starlike function of complex order b, $b \in \mathbb{C} \setminus \{0\}$, if and only if $\frac{f(z)}{z} \neq 0$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \qquad (z \in \mathcal{U}) \qquad (2.4.5)$$

The class of all starlike functions of complex order b, $b \in \mathbb{C} \setminus \{0\}$ is denoted by S_b^*. Notice that,

$$S_1^* = S^*, S_{1-\alpha}^* = S^*(\alpha); 0 \leq \alpha < 1$$

In 1982, Nasr and Aouf [35] introduced the following subclass of convex functions in the open unit disk, that is K_b.

Definition 2.4.12

A function $f(z) \in S$ is said to be convex function of complex order $b, b \in \mathbb{C} \setminus \{0\}$, if and only if $f'(z) \neq 0$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathcal{U}) \qquad (2.4.6)$$

The class of all convex functions of complex order $b, b \in \mathbb{C} \setminus \{0\}$ is denoted by K_b. Notice that,

$$K_1 = K, K_{1-\alpha} = K(\alpha); 0 \leq \alpha < 1$$

2.5 Close-to-convex functions

In this section, other well-known subclass of univalent functions in the open unit disk, namely the close-to-convex functions is considered.

In 1952, Kaplan [25] introduced the class of close-to-convex functions as follows.

Definition 2.5.1

A function $f(z) \in S$ is said to be close-to-convex function if there is a convex function g such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \qquad (z \in \mathcal{U}) \qquad (2.5.1)$$

An equivalent formulation would involve the existence of a starlike function $h(z)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{h(z)}\right) > 0, \qquad (z \in \mathcal{U}) \qquad (2.5.2)$$

The class of all close-to-convex functions is denoted by $\mathcal{C}$.

There is a beautiful relationship between close-to-convex functions and univalent functions given by the following theorem.

Theorem 2.5.2 [15]

Every close-to-convex function is univalent.

Remark 2.5.1

Every convex function is close-to-convex and every starlike function is close-to-convex. Then it is clear that $K \subset S^* \subset \mathcal{C} \subset S$.

In 1956, Reade [50] introduced the following subclass of close-to-convex functions in the open unit disk, that is $\mathcal{C}(\alpha)$.

Definition 2.5.3

A function $f(z) \in S$ is said to be close-to-convex function of order $\alpha, 0 \leq \alpha < 1$ if there is a convex function g such that

$$\text{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha \,, \qquad (z \in \mathcal{U}) \qquad (2.5.3)$$

An equivalent formulation would involve the existence of a starlike function $h(z)$ such that

$$\text{Re}\left(\frac{zf'(z)}{h(z)}\right) > \alpha, \qquad (z \in \mathcal{U}) \qquad (2.5.4)$$

The class of all close-to-convex functions of order $\alpha, 0 \leq \alpha < 1$ is denoted by $\mathcal{C}(\alpha)$. Notice that,

$$\mathcal{C}(0) = \mathcal{C}$$

2.6 Multivalent starlike and convex functions

This section is devoted to study the most important subclasses of the class $A(p)$ of p-valent (or multivalent) functions, namely the classes of p-valent starlike and p-valent convex functions which were studied by Goodman [17] and defined as follows.

Definition 2.6.1

A function $f(z) \in A(p)$ is said to be p-valent starlike function if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (p \in \mathbb{N};\ z \in \mathcal{U}) \qquad (2.6.1)$$

The class of all p-valent starlike functions is denoted by $S^*(p)$. Notice that,

$$S^*(1) = S^*$$

Definition 2.6.2

A function $f(z) \in A(p)$ is said to be p-valent convex function if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.6.2)$$

The class of all p-valent convex functions is denoted by $K(p)$. Notice that

$$K(1) = K$$

Patil and Thakare [44] introduced the following subclass of p-valent starlike functions in the open unit disk, that is $S^*(p,\alpha)$.

Definition 2.6.3

A function $f(z) \in A(p)$ is said to be p-valent starlike function of order $\alpha, 0 \leq \alpha < p$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (p \in \mathbb{N};\ z \in \mathcal{U}) \qquad (2.6.3)$$

The class of all p-valent starlike functions of order $\alpha, 0 \leq \alpha < p$ is denoted by $S^*(p,\alpha)$. Notice that,

$$S^*(p,0) = S^*(p),\ S^*(1,\alpha) = S^*(\alpha),\ S^*(1,0) = S^*$$

Owa [40] introduced the following subclass of p-valent convex functions in the open unit disk, that is $K(p,\alpha)$.

Definition 2.6.4

A function $f(z) \in A(p)$ is said to be p-valent convex function of order $\alpha, 0 \leq \alpha < p$ if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.6.4)$$

The class of all p-valent convex functions of order $\alpha, 0 \leq \alpha < p$ is denoted by $K(p,\alpha)$. Notice that,

$$K(p,0) = K(p), K(1,\alpha) = K(\alpha),\ K(1,0) = K$$

There is an Alexander type result [40] relating $S^*(p,\alpha)$ and $K(p,\alpha)$ which says the function $f(z) \in K(p,\alpha)$ if and only if $zf'(z)/p \in S^*(p,\alpha)$ for $0 \leq \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$.

The following subclass of p-valent starlike functions in the open unit disk, that is $S^*_{b,p}$ was given by El Ashwah [16] as follows.

Definition 2.6.5

A function $f(z) \in A(p)$ is said to be p-valent starlike function of complex order $b, b \in \mathbb{C} \setminus \{0\}$, if and only if $\frac{f(z)}{z} \neq 0$ and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{pf(z)} - 1\right)\right\} > 0, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.6.5)$$

The class of all p-valent starlike functions of complex order $b, b \in \mathbb{C} \setminus \{0\}$ is denoted by $S^*_{b,p}$. Notice that,

$$S^*_{b,1} = S^*_b,\ S^*_{1,1} = S^*$$

Aouf [9] introduced the following subclass of p-valent convex functions in the open unit disk, that is $K_{b,p}$.

Definition 2.6.6

A function $f(z) \in A(p)$ is said to be p-valent convex function of complex order $b, b \in \mathbb{C} \setminus \{0\}$, if and only if $f'(z) \neq 0$ and

$$\mathrm{Re}\left\{1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.6.6)$$

The class of all p-valent convex functions of complex order $b, b \in \mathbb{C} \setminus \{0\}$ is denoted by $K_{b,p}$. Notice that,

$$K_{b,1} = K_b, K_{1,1} = K$$

2.7 Multivalent close-to-convex functions

In this section, other well-known subclass of p-valent functions in the open unit disk, namely the close-to-convex functions is defined.

Livingston [30] defined the class of p-valent close-to-convex functions as follows.

Definition 2.7.1

A function $f(z) \in A(p)$ is said to be p-valent close-to-convex function if there is a convex function $g(z)$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.7.1)$$

An equivalent formulation would involve the existence of a starlike function $h(z)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{h(z)}\right) > 0 \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.7.2)$$

The class of all p-valent close-to-convex functions is denoted by $\mathcal{C}(p)$. Notice that,

$$\mathcal{C}(1) = \mathcal{C}$$

Also, Mishra and Sahu [34] introduced the following subclass of p-valent close-to-convex functions in the open unit disk, that is $\mathcal{C}(p, \alpha)$.

Definition 2.7.2

A function $f(z) \in A(p)$ is said to be p-valent close-to-convex function of order $\alpha, 0 \leq \alpha < p$ if there is a convex function $g(z)$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.7.3)$$

An equivalent formulation would involve the existence of a starlike function $h(z)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{h(z)}\right) > \alpha, \qquad (p \in \mathbb{N}; z \in \mathcal{U}) \qquad (2.7.4)$$

The class of all p-valent close-to-convex functions of order $\alpha, 0 \leq \alpha < p$ is denoted by $\mathcal{C}(p, \alpha)$. Notice that,

$$\mathcal{C}(p,0) = \mathcal{C}(p), \mathcal{C}(1,\alpha) = \mathcal{C}(\alpha), \mathcal{C}(1,0) = \mathcal{C}$$

2.8 Linear operators

In this section, the definitions of certain known differential and integral operators are introduced, which will be required in later sections.

2.8.1 Sălăgean differential operators

In 1983, Sălăgean [55] defined and studied the following differential operator for $f(z) \in A$.

Definition 2.8.1.1

For $f(z) \in A$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the differential operator $D^m : A \to A$ is defined by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

and (in general)

$$\begin{aligned} D^m f(z) &= D\left(D^{m-1} f(z)\right) \\ &= z + \sum_{n=2}^{\infty} n^m a_n z^n, \qquad (z \in \mathcal{U}) \end{aligned} \tag{2.8.1.1}$$

Motivated essentially by Sălăgean [55], the p-valent Sălăgean differential operator for $f(z) \in A(p)$ was given by Shenan et. al. [59] as follows.

Definition 2.8.1.2

For $f(z) \in A(p), m \in \mathbb{N}_0$ and $p \in \mathbb{N}$, the differential operator $D_p^m : A(p) \to A(p)$ is defined by

$$D_p^0 f(z) = f(z)$$

$$D_p^1 f(z) = D_p f(z) = \frac{1}{p} zf'(z)$$

and (in general)

$$D_p^m f(z) = D_p(D_p^{m-1} f(z))$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right)^m a_{p+n} z^{p+n}, \qquad (z \in \mathcal{U}) \qquad (2.8.1.2)$$

Notice that,

$$D_1^m f(z) = D^m f(z)$$

2.8.2 Al-Oboudi differential operators

In 2004, Al-Oboudi [3] defined and studied the following differential operator for $f(z) \in A$.

Definition 2.8.2.1

For $f(z) \in A, m \in \mathbb{N}_0$ and $\delta \geq 0$, the differential operator $D_\delta^m : A \to A$ is defined by

$$D_\delta^0 f(z) = f(z)$$

$$D_\delta^1 f(z) = D_\delta f(z) = (1-\delta) f(z) + \delta z f'(z)$$

and (in general)

$$D_\delta^m f(z) = D_\delta(D_\delta^{m-1} f(z))$$

$$= z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]^m a_n z^n, \qquad (z \in \mathcal{U}) \qquad (2.8.2.1)$$

Notice that,

$$D_1^m f(z) = D^m f(z)$$

Motivated essentially by Al-Oboudi [3], the p-valent Al-Oboudi differential operator for $f(z) \in A(p)$ was defined by Aouf [10] as follows.

Definition 2.8.2.2

For $f(z) \in A(p)$, $m \in \mathbb{N}_0$, $\delta \geq 0$ and $p \in \mathbb{N}$, the differential operator $D_{\delta,p}^m : A(p) \to A(p)$ is defined by

$$D_{\delta,p}^{0} f(z) = f(z)$$

$$D_{\delta,p}^{1} f(z) = D_{\delta,p} f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z)$$

and (in general)

$$D_{\delta,p}^{m} f(z) = D_{\delta,p}(D_{\delta,p}^{m-1} f(z))$$

$$= z^{p} + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^{m} a_{p+n} z^{p+n}, \qquad (z \in \mathcal{U}) \qquad (2.8.2.2)$$

Notice that,

$$D_{1,p}^{m} f(z) = D_{p}^{m} f(z), D_{\delta,1}^{m} f(z) = D_{\delta}^{m} f(z)$$

2.8.3 Fractional derivative operators

The following fractional derivative operators $D_{0,z}^{\lambda} f(z)$ and $J_{0,z}^{\lambda,\mu,\eta} f(z)$ were given by Owa [39] and Raina and Nahar [47], respectively.

Definition 2.8.3.1

Let $0 \le \lambda < 1$, the fractional derivative operator of order λ is defined by

$$D_{0,z}^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \qquad (2.8.3.1)$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2.8.3.2 [62]

The Gauss hypergeometric function is denoted by ${}_2F_1(a,b,c;z)$ and defined by

$$ {}_2F_1(a,b,c;\, z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!} z^{n}, \qquad (z \in \mathcal{U})$$

where $(\lambda)_n$ is the Pochhammer symbol given in terms of the Gamma function Γ by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1\,, & n=0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n\in\mathbb{N}. \end{cases}$$

for $\lambda \neq 0, -1, -2, \ldots$.

Definition 2.8.3.3

Let $0 \le \lambda < 1$, and $\mu, \eta \in \mathbb{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \times \right.$$

$$\left. \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta, 1-\lambda; 1-\frac{\xi}{z}\right) d\xi \right) \qquad (2.8.3.2)$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Owa and Srivastava [43] defined the fractional derivative operator by making use of the operator $D_{0,z}^{\lambda} f(z)$ given by (2.8.3.1) for $f(z) \in A$ as follows.

Definition 2.8.3.4

For $f(z) \in A$ and $0 \le \lambda \le 1$, the fractional derivative operator $\Omega_{0,z}^{\lambda} f(z)$ is defined by

$$\Omega_{0,z}^{\lambda} f(z) = \Gamma(2-\lambda)\, z^{\lambda}\, D_{0,z}^{\lambda} f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n\,, \qquad (z \in \mathcal{U}) \qquad (2.8.3.3)$$

where $D_{0,z}^{\lambda} f(z)$ is given by (2.8.3.1). Notice that,

$$\Omega_{0,z}^{0} f(z) = f(z),\ \Omega_{0,z}^{1} f(z) = z f'(z)$$

Motivated by Owa and Srivastava [43] and by making use of the operator $J_{0,z}^{\lambda,\mu,\eta}f(z)$ given by (2.8.3.2), Raina and Nahar [47] introduced the generalized fractional derivative operator for $f(z) \in A$ as follows.

Definition 2.8.3.5

For $f(z) \in A$, $\lambda \geq 0$, $\mu < 2$ and $\eta > \max(\lambda, \mu) - 2$, the fractional derivative operator $P_{0,z}^{\lambda,\mu,\eta}f(z)$ is defined by

$$P_{0,z}^{\lambda,\mu,\eta}f(z) = \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta}f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(2)_{n-1}(2-\mu+\eta)_{n-1}}{(2-\mu)_{n-1}(2-\lambda+\eta)_{n-1}} a_n z^n, \quad (z \in \mathcal{U}) \qquad (2.8.3.4)$$

where $J_{0,z}^{\lambda,\mu,\eta}f(z)$ is given by (2.8.3.2). Notice that,

$$P_{0,z}^{\lambda,\lambda,\eta}f(z) = \Omega_{0,z}^{\lambda}f(z)$$

Motivated essentially by the above works, a more general fractional derivative operator was studied by Amsheri and Zharkova [7], [8] for $f(z) \in A(p)$ as follows.

Definition 2.8.3.6

For $f(z) \in A(p), \lambda \geq 0, \mu < p+1, \eta > \max(\lambda, \mu) - p - 1$ and $p \in \mathbb{N}$, the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p}f(z)$ is defined by

$$M_{0,z}^{\lambda,\mu,\eta,p}f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta}f(z)$$

$$= z^p + \sum_{n=1}^{\infty} \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n}, \qquad (z \in \mathcal{U}) \qquad (2.8.3.5)$$

where $J_{0,z}^{\lambda,\mu,\eta}f(z)$ is given by (2.8.3.2) and

$$\gamma_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}, \qquad (n \in \mathbb{N}) \qquad (2.8.3.6)$$

Notice that,

$$M_{0,z}^{\lambda,\mu,\eta,1} f(z) = P_{0,z}^{\lambda,\mu,\eta} f(z)\ ,\ M_{0,z}^{0,0,\eta,p} f(z) = f(z) \text{ and } M_{0,z}^{1,1,\eta,p} f(z) = \frac{zf'(z)}{p}$$

Very recently, Zayed et. al. [64] defined and studied the generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ for $f(z) \in A(p)$ based on Al-Oboudi differential operator and fractional derivative operator as follows.

Definition 2.8.3.7

For $f(z) \in A(p), m \in \mathbb{N}_0, \lambda \geq 0, \mu < p+1, \eta > \max(\lambda,\mu) - p - 1, \delta \geq 0$, $z \in \mathcal{U}$, $p \in \mathbb{N}$ and $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ is given by (2.8.3.5), the generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ is defined by

$$N_{0,z}^{0,\lambda,\mu,\eta,\delta,p} f(z) = M_{0,z}^{\lambda,\mu,\eta,p} f(z)$$

$$\begin{aligned} N_{0,z}^{1,\lambda,\mu,\eta,\delta,p} f(z) &= N_{0,z}^{\lambda,\mu,\eta,\delta,p} f(z) \\ &= (1-\delta) M_{0,z}^{\lambda,\mu,\eta,p} f(z) + \delta \frac{z}{p} \left(M_{0,z}^{\lambda,\mu,\eta,p} f(z) \right)' \\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right) \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \end{aligned}$$

and (in general)

$$\begin{aligned} N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) &= N_{0,z}^{\lambda,\mu,\eta,\delta,p} (N_{0,z}^{m-1,\lambda,\mu,\eta,\delta,p} f(z)) \\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right)^m \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \end{aligned}$$

(2.8.3.7)

Similarly, for $f(z) \in T(p)$, we can write $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ in the form

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) = z^p - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right)^m \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n}, \quad (a_{p+n} \geq 0)$$

(2.8.3.8)

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). Notice that,

$$N_{0,z}^{m,\lambda,\mu,\eta,0,p} f(z) = M_{0,z}^{\lambda,\mu,\eta,p} f(z), N_{0,z}^{m,0,0,\eta,\delta,p} f(z) = D_{\delta,p}^{m} f(z)$$

$$N_{0,z}^{m,0,0,\eta,0,p} f(z) = f(z), N_{0,z}^{m,1,1,\eta,0,p} f(z) = \frac{zf'(z)}{p}$$

From (2.8.3.7), we have

$$z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)' = (p-\mu) N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p} f(z) + \mu N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \qquad (2.8.3.9)$$

2.8.4 Integral operators

The following integral operator $J_c\big(f(z)\big)$ for $f(z) \in A$ was introduced by Bernardi [11] and known as Bernardi-Libera-Livingston integral operator. It generalizes the integral operator due to Libera [27] and Livingston [31].

Definition 2.8.4.1

For $f(z) \in A$ and $c > -1$, the integral operator $J_c(f(z))$ is defined by

$$J_c\big(f(z)\big) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

$$= z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n , \qquad (z \in \mathcal{U}) \qquad (2.8.4.1)$$

Motivated essentially by the above works, Saitoh et. al. [57] introduced the generalized Bernardi-Libera-Livingston integral operator $J_{c,p}\big(f(z)\big)$ for $f(z) \in A(p)$ as follows.

Definition 2.8.4.2

For $f(z) \in A(p)$, $c > -p$ and $p \in \mathbb{N}$, the integral operator $J_{c,p}\big(f(z)\big)$ is defined by

$$J_{c,p}\big(f(z)\big) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right) a_{p+n}\, z^{p+n} , \qquad (z \in \mathcal{U}) \qquad (2.8.4.2)$$

Chapter 3

Fekete-Szegö inequalities for certain classes of analytic functions

The main objective of this chapter is to obtain coefficient bounds for the functional $\left|a_{p+2}-\theta a_{p+1}^2\right|$ of functions belonging to certain classes of analytic and p-valent functions defined in the open unit disk which generalized the concept of starlike functions.

3.1 Introduction and preliminaries

Let $\mathcal{P}$ is the class of all analytic functions with a positive real part in the open unit disk defined by

$$p(z)=1+\sum_{n=1}^{\infty} c_n z^n$$

with $p(0)=1$ and $\operatorname{Re}p(z)>0$, $z\in\mathcal{U}$. It is well-known that $|c_n|\leq 2$ $(n=1,2,\dots)$ [46].

In 1933, Fekete and Szegö obtained the sharp bound for $|a_3-\eta a_2^2|$ as a function of the real parameter η and proved that

$$|a_3-\eta a_2^2|\leq 1+2\exp\left(-\frac{2\eta}{1-\eta}\right), \qquad (0\leq\eta\leq 1)$$

for functions in the class S [13]. Later, the problem of finding sharp bound for the functional $|a_3-\eta a_2^2|$ of any compact family of functions $f(z)\in A$ is known as the Fekete-Szegö problem or inequality.

In 1994, Ma and Minda [32] gave an unified treatment of various subclasses consisting of starlike and convex functions for which either the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function which defined as follows.

Definition 3.1.1

Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0)=1$ and $\phi'(0)>0$. A function $f(z)\in S$ is said to be in the class $S^*(\phi)$ if

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z\in\mathcal{U}) \qquad (3.1.1)$$

and $C(\phi)$ be the class of functions $f(z)\in S$ for which

$$1+\frac{zf''(z)}{f'(z)} \prec \phi(z), \qquad (z\in\mathcal{U}) \qquad (3.1.2)$$

A function $f(z)\in S^*(\phi)$ is said to be starlike with respect to the function ϕ, and a function $f(z)\in C(\phi)$ is said to be convex with respect to the function ϕ.

For these classes, the estimates for the first few coefficients and Fekete-Szegö inequalities have been obtained [32].

Following Ma and Minda [32], Ravichandran et. al. [49] defined a more general classes related to the classes of starlike and convex functions of complex order as follows.

Definition 3.1.2

Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0)=1$ and $\phi'(0)>0$. A function $f(z)\in A$ is said to be in the class $S_b^*(\phi)$ if

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right) \prec \phi(z), \qquad (z\in\mathcal{U}) \qquad (3.1.3)$$

and $C_b(\phi)$ be the class of functions $f(z)\in A$ for which

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z), \qquad (z \in \mathcal{U}) \qquad (3.1.4)$$

where $b \in \mathbb{C}\backslash\{0\}$.

Notice that, for $b = 1$, the class $S_1^*(\phi)$ is the class $S^*(\phi)$ and the class $C_1(\phi)$ is the class $C(\phi)$.

For the classes $S_b^*(\phi)$ and $C_b(\phi)$, the necessary and sufficient conditions for functions to belong to these classes and Fekete-Szegö inequalities have been obtained [49].

In this chapter, motivated by a-fore-mentioned works and by linear operators which were studied by [8], [48] and others, Fekete-Szegö inequalities for well-known classes of p-valent functions are obtained. Furthermore, a new class of p-valent functions associated with generalized differential operator is introduced and Fekete-Szegö inequalities are obtained.

Now, in order to prove the results in the current chapter, the following lemma given by Ali et. al. [2] is needed.

Lemma 3.1.3

If $w(z) \in \Omega$, then

$$|w_2 - tw_1^2| \leq \begin{cases} -t, & t \leq -1, \\ 1, & -1 \leq t \leq 1, \\ t, & t \geq 1, \end{cases}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}$, $0 \leq \lambda \leq 1$ or one of its rotations, while for $t = 1$, the equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$, $0 \leq \lambda \leq 1$ or one of its rotations. Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \leq 1\,, \qquad (-1 < t \leq 0)$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leq 1\,, \qquad (0 < t < 1)$$

Further, the following lemma given by Keogh and Merkes [26] is needed as well.

Lemma 3.1.4

If $w(z) \in \Omega$, then for any complex number t,

$$|w_2 - tw_1^2| \leq \max(1, |t|)$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

3.2 Certain class of p-valent functions

In 2007, Ali et. al. [2] extended the classes $S^*(\phi)$ and $C(\phi)$ of univalent functions which were introduced by Ma and Minda [32] as well as the classes $S_b^*(\phi)$ and $C_b(\phi)$ of univalent functions of complex order which were defined by Ravichandran et. al. [49] to more general classes of p-valent functions in order to obtain Fekete-Szegö inequalities. Authors [2] have defined the class $S_{b,p}^*(\phi)$ of p-valent functions of complex order with respect to the function ϕ as follows.

Definition 3.2.1

Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in A(p)$ is said to be in the class $S_{b,p}^*(\phi)$ if

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{pf(z)} - 1\right) \prec \phi(z), \qquad (z \in \mathcal{U}) \qquad (3.2.1)$$

where $b \in \mathbb{C}\backslash\{0\}$ and $p \in \mathbb{N}$. Further, let $S_{1,p}^*(\phi) = S_p^*(\phi)$.

It may be noted that the class $S_{b,p}^*(\phi)$ extends the class of starlike functions for suitable choice of b and p. In particular, for $p = 1$, the a-fore-mentioned class reduces to the class $S_b^*(\phi)$ which was introduced and studied by Ravichandran et. al. [49]. For $b = 1$ and $p = 1$, it reduces to the class $S^*(\phi)$ which was introduced by Ma and Minda [32].

Now, by making use of the Lemma 3.1.3 and Lemma 3.1.4, the coefficient bounds for functions belonging to the class $S_p^*(\phi)$ according to Ali et. al. [2] are obtained as follows.

Theorem 3.2.2

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n are real with $B_1 > 0$, $B_2 \geq 0$, and

$$\sigma_1 = \frac{B_2 - B_1 + pB_1^2}{2pB_1^2},$$

$$\sigma_2 = \frac{B_2 + B_1 + pB_1^2}{2pB_1^2},$$

$$\sigma_3 = \frac{B_2 + pB_1^2}{2pB_1^2}.$$

If $f(z) \in A(p)$ belongs to $S_p^*(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \begin{cases} \dfrac{p}{2}\left(B_2 + (1-2\theta)pB_1^2\right), & \theta \leq \sigma_1, \\ \dfrac{pB_1}{2}, & \sigma_1 \leq \theta \leq \sigma_2, \\ -\dfrac{p}{2}\left(B_2 + (1-2\theta)pB_1^2\right), & \theta \geq \sigma_2, \end{cases} \tag{3.2.2}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 - \frac{B_2}{B_1} + (2\theta - 1)pB_1\right)\left|a_{p+1}\right|^2 \leq \frac{pB_1}{2} \tag{3.2.3}$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 + \frac{B_2}{B_1} - (2\theta - 1)pB_1\right)\left|a_{p+1}\right|^2 \leq \frac{pB_1}{2} \tag{3.2.4}$$

For any complex number θ,

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \frac{pB_1}{2} \max\left\{1, \left|\frac{B_2}{B_1} + (1-2\theta)pB_1\right|\right\} \qquad (3.2.5)$$

The results are sharp.

Proof

If $f(z) \in S_p^*(\phi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$$

such that

$$\frac{zf'(z)}{pf(z)} = \phi\big(w(z)\big) \qquad (3.2.6)$$

since

$$\frac{zf'(z)}{pf(z)} = 1 + \frac{1}{p} a_{p+1} z + \frac{1}{p}\left[2a_{p+2} - a_{p+1}^2\right] z^2 + \cdots$$

we have from (3.2.6),

$$a_{p+1} = p\, B_1 w_1 \qquad (3.2.7)$$

and

$$a_{p+2} = \frac{1}{2}\,\{pB_1 w_2 + p(B_2 + pB_1^2) w_1^2\} \qquad (3.2.8)$$

Using (3.2.7) and (3.2.8), we have

$$a_{p+2} - \theta a_{p+1}^2 = \frac{pB_1}{2}\,\{w_2 - v w_1^2\}$$

where

$$v = p(2\theta - 1)B_1 - \frac{B_2}{B_1}$$

The result (3.2.2)-(3.2.4) are established by an application of Lemma 3.1.3 and the inequality (3.2.5) by Lemma 3.1.4. To show that the bounds in (3.2.2)-(3.2.4) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3, \ldots)$ by

$$\frac{zK'_{\phi n}(z)}{pK_{\phi n}(z)} = \phi(z^{n-1}), \qquad \left(K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0\right)$$

and the functions F_r, G_r $(0 \leq r \leq 1)$ defined by

$$\frac{zF_r'(z)}{pF_r(z)} = \phi\left(\frac{z(z+r)}{1+rz}\right), \qquad (F_r(0) = F_r{}'(0) - 1 = 0\ )$$

and

$$\frac{zG_r'(z)}{pG_r(z)} = \phi\left(-\frac{z(z+r)}{1+rz}\right), \qquad (G_r(0) = G_r{}'(0) - 1 = 0)$$

clearly the functions $K_{\phi n}, F_r,\ G_r \in S_p^*(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. If $\sigma_1 < \theta < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations.

Remark 3.2.1

Letting $p = 1$ in Theorem 3.2.2, the following result due to Ma and Minda [32] is obtained.

Corollary 3.2.3

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n are real with $B_1 > 0,\ B_2 \geq 0$, and

$$\sigma_1 = \frac{B_2 - B_1 + B_1^2}{2B_1^2},$$

$$\sigma_2 = \frac{B_2 + B_1 + B_1^2}{2B_1^2}.$$

If $f(z) \in S$ belongs to $S^*(\phi)$, then

$$|a_3 - \theta a_2^2| \leq \begin{cases} \dfrac{B_2 + (1 - 2\,\theta)B_1^2}{2}, & \theta \leq \sigma_1, \\ \dfrac{B_1}{2}, & \sigma_1 \leq \theta \leq \sigma_2, \\ -\dfrac{B_2 + (1 - 2\,\theta)B_1^2}{2}, & \theta \geq \sigma_2\,, \end{cases} \tag{3.2.9}$$

3.3 Certain class of p-valent functions associated with fractional derivative operator

This section refers to some applications of the generalized fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ defined by (2.8.3.5) in order to obtain Fekete-Szegö inequalities. In 2012, Amsheri and Zharkova [8] extended the classes $S_{b,p}^{*}(\phi)$ [2], $S_{b}^{*}(\phi)$ [49] and $S^{*}(\phi)$ [32] to more general class of p-valent functions of complex order associated with the operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$. Authors [8] have defined the class $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$ of p-valent functions of complex order with respect to the function ϕ as follows.

Definition 3.3.1

Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in A(p)$ is said to be in the class $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$ if

$$1 + \frac{1}{b}\left(\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)}{M_{0,z}^{\lambda,\mu,\eta,p} f(z)} - 1\right) \prec \phi(z), \quad (z \in \mathcal{U}) \tag{3.3.1}$$

where $\lambda \geq 0$, $\mu < p + 1, \eta > \max(\lambda, \mu) - p - 1$, $b \in \mathbb{C}\backslash\{0\}$ and $p \in \mathbb{N}$. Further, let $S_{1,p,\lambda,\mu,\eta}^{*}(\phi) = S_{p,\lambda,\mu,\eta}^{*}(\phi)$.

The above class $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$ is of special interest and it contains many well-known classes of analytic functions. In particular, for $b = p = 1$ and $\lambda = \mu$ =0, the a-fore-mentioned class reduces to the class S^{*} (ϕ) which was investigated by Ma and Minda [32]. For $p = 1$ and $\lambda = \mu$ =0, it reduces to the class S_{b}^{*} (ϕ) which was studied by Ravichandran et. al. [49]. For $b = 1$ and $\lambda = \mu$ =0, it reduces to the class $S_{p}^{*}(\phi)$ which was studied by Ali et. al. [2].

Now, by making use of Lemma 3.1.3 and Lemma 3.1.4, the coefficient bounds for functions belonging to the class $S^*_{p,\lambda,\mu.\eta}(\phi)$ according to Amsheri and Zharkova [8] are obtained as follows.

Theorem 3.3.2

Let $0 \le \theta \le 1, \lambda \ge 0, \mu < p+1, \eta > \max(\lambda, \mu) - p - 1$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n are real with $B_1 > 0$, $B_2 \ge 0$, and

$$\sigma_1 = \frac{(B_2 - B_1)\gamma_1^2 + (p-\mu)B_1^2\gamma_1^2}{2\gamma_2 B_1^2 (p-\mu)},$$

$$\sigma_2 = \frac{(B_2 + B_1)\gamma_1^2 + (p-\mu)B_1^2\gamma_1^2}{2\gamma_2 B_1^2 (p-\mu)},$$

$$\sigma_3 = \frac{B_2\gamma_1^2 + (p-\mu)B_1^2\gamma_1^2}{2\gamma_2 B_1^2 (p-\mu)}.$$

If $f(z) \in A(p)$ belongs to $S^*_{p,\lambda,\mu,\eta}(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \le \begin{cases} \dfrac{(p-\mu)}{2\gamma_2}\left(B_2 - \dfrac{(p-\mu)(2\gamma_2\theta - \gamma_1^2)}{\gamma_1^2} B_1^2\right), & \theta \le \sigma_1, \\ \dfrac{(p-\mu)B_1}{2\gamma_2}, & \sigma_1 \le \theta \le \sigma_2, \\ -\dfrac{(p-\mu)}{2\gamma_2}\left(B_2 - \dfrac{(p-\mu)(2\gamma_2\theta - \gamma_1^2)}{\gamma_1^2} B_1^2\right), & \theta \ge \sigma_2, \end{cases} \tag{3.3.2}$$

Further, if $\sigma_1 \le \theta \le \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{\gamma_1^2}{2\gamma_2 B_1 (p-\mu)} \times \left(1 - \frac{B_2}{B_1} + \frac{(p-\mu)(2\gamma_2\theta - \gamma_1^2)}{\gamma_1^2} B_1\right)\left|a_{p+1}\right|^2 \le \frac{(p-\mu)B_1}{2\gamma_2} \tag{3.3.3}$$

If $\sigma_3 \le \theta \le \sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right| + \frac{\gamma_1^2}{2\gamma_2 B_1(p-\mu)}$$

$$\times\left(1+\frac{B_2}{B_1}-\frac{(p-\mu)(2\gamma_2\theta-\gamma_1^2)}{\gamma_1^2}B_1\right)\left|a_{p+1}\right|^2 \le \frac{(p-\mu)B_1}{2\gamma_2} \tag{3.3.4}$$

For any complex number θ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right| \le \frac{(p-\mu)B_1}{2\gamma_2}\max\left\{1,\left|\frac{(p-\mu)(2\gamma_2\theta-\gamma_1^2)}{\gamma_1^2}B_1-\frac{B_2}{B_1}\right|\right\} \tag{3.3.5}$$

The results are sharp.

Proof

If $f(z) \in S_{p,\lambda,\mu,\eta}^{*}(\phi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$$

such that

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)}{M_{0,z}^{\lambda,\mu,\eta,p} f(z)} = \phi\big(w(z)\big) \tag{3.3.6}$$

since

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)}{M_{0,z}^{\lambda,\mu,\eta,p} f(z)} = 1 + \frac{\gamma_1}{p-\mu}a_{p+1}z + \frac{1}{p-\mu}\left[2\gamma_2 a_{p+2} - \gamma_1^2 a_{p+1}^2\right]z^2 + \cdots$$

we have from (3.3.6)

$$a_{p+1} = \frac{(p-\mu)w_1}{\gamma_1} B_1 \tag{3.3.7}$$

and

$$a_{p+2} = \frac{(p-\mu)}{2\gamma_2}\{B_1 w_2 + (B_2 + (p-\mu)B_1^2)w_1^2\} \tag{3.3.8}$$

Therefore, we have

$$a_{p+2} - \theta a_{p+1}^2 = \frac{(p-\mu)B_1}{2\gamma_2}\ \{w_2 - vw_1^2\}$$

where

$$v = \frac{(p-\mu)B_1(2\gamma_2\theta - \gamma_1^2)}{\gamma_1^2} - \frac{B_2}{B_1}$$

The results (3.3.2)-(3.3.4) are established by an application of Lemma 3.1.3 and inequality (3.3.5) by Lemma 3.1.4. To show that the bounds in (3.3.2)-(3.3.4) are sharp, we define the functions $K_{\phi n}(z)$ $(n = 2,3,\dots)$ by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} K_{\phi n}(z)}{M_{0,z}^{\lambda,\mu,\eta,p} K_{\phi n}(z)} = \phi(z^{n-1}), \qquad \left(K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0\right)$$

and the functions F_r, G_r $(0 \leq r \leq 1)$ defined by

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} F_r(z)}{M_{0,z}^{\lambda,\mu,\eta,p} F_r(z)} = \phi\left(\frac{z(z+r)}{1+rz}\right), \qquad (F_r(0) = F_r'(0) - 1 = 0)$$

and

$$\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1,p} G_r(z)}{M_{0,z}^{\lambda,\mu,\eta,p} G_r(z)} = \phi\left(-\frac{z(z+r)}{1+rz}\right), \qquad (G_r(0) = G_r'(0) - 1 = 0)$$

respectively, it is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $S_{p,\lambda,\mu,\eta}^*(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. If $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations.

Remark 3.3.1

By taking $\lambda = \mu = 0$ in Theorem 3.3.2, Theorem 3.2.2 due to Ail et. al. [2] is obtained.

Remark 3.3.2

By taking $\lambda = \mu = 0$ and $p = 1$ in Theorem 3.3.2, Corollary 3.2.3 due to Ma and Minda [32] is obtained.

In the similar manner, the coefficient bound for $|a_{p+2} - \theta a_{p+1}^2|$ of functions in the class $S^*_{b,p,\lambda,\mu,\eta}(\phi)$ [8] is given as follows.

Theorem 3.3.3

Let $0 \leq \theta \leq 1, \lambda \geq 0, \mu < p+1, \eta > \max(\lambda, \mu) - p - 1, p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z) \in A(p)$ belongs to $S^*_{b,p,\lambda,\mu,\eta}(\phi)$, then for any complex number θ, we have

$$|a_{p+2} - \theta a_{p+1}^2| \leq \frac{(p-\mu)|b|B_1}{2\gamma_2} \max\left\{1, \left|\frac{(p-\mu)b(2\gamma_2\theta - \gamma_1^2)}{\gamma_1^2} B_1 - \frac{B_2}{B_1}\right|\right\} \tag{3.3.9}$$

The result is sharp.

Remark 3.3.3

By taking $\lambda = \mu = 0$ and $p = 1$ in Theorem 3.3.3, the corresponding result due to Ravichandran et. al. [49] is obtained.

3.4 Certain class of p-valent functions associated with generalized differential operator

Motivated by the a-fore-mentioned works in the current chapter, a new class $S^m_{b,p,\lambda,\mu,\eta,\delta}(\phi)$ of p-valent starlike functions of complex order associated with generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ defined by (2.8.3.7) is introduced and Fekete-Szegö inequalities are obtained according to Amsheri and Abouthfeerah [4].

Definition 3.4.1

Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis such that $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f(z) \in A(p)$ is said to be in the class $S^m_{b,p,\lambda,\mu,\eta,\delta}(\phi)$ if

$$1+\frac{1}{b}\left(\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-1\right)\prec\phi(z), \qquad (z\in\mathcal{U}) \qquad (3.4.1)$$

where $m\in\mathbb{N}_0$, $\lambda\geq 0$, $\mu<p+1, \eta>\max(\lambda,\mu)-p-1, \delta\geq 0, b\in\mathbb{C}\backslash\{0\}$ and $p\ \in\mathbb{N}$. Further, let $S_{1,p,\lambda,\mu,\eta,\delta}^{m}(\phi)=S_{p,\lambda,\mu,\eta,\delta}^{m}(\phi)$.

The above class $S_{b,p,\lambda,\mu,\eta,\delta}^{m}(\phi)$ is of special interest and it contains many well-known classes of analytic functions. In particular, for $b=p=1$ and $\lambda=\mu=\delta=0$, the a-fore-mentioned class reduces to the class $S^{*}(\phi)$ which was investigated by Ma and Minda [32]. For $p=1$ and $\lambda=\mu=\delta=0$, it reduces to the class $S_b^*(\phi)$ which was studied by Ravichandran et. al. [49]. For $b=1$ and $\lambda=\mu=\delta=0$, it reduces to the class $S_p^*(\phi)$ which was studied by Ali et. al. [2]. Furthermore, when $m=0$, it reduces to the class $S_{b,p,\lambda,\mu,\eta}^{*}(\phi)$ which was introduced by Amsheri and Zharkova [8].

Now, by making use of Lemma 3.1.3 and Lemma 3.1.4, the coefficient bounds for functions belonging to the class $S_{p,\lambda,\mu,\eta,\delta}^{m}(\phi)$ [4] are obtained as follows.

Theorem 3.4.2

Let $0\leq\theta\leq 1, m\in\mathbb{N}_0, \lambda\geq 0, \mu<p+1, \eta>\max(\lambda,\mu)-p-1, \delta\geq 0$ and $p\in\mathbb{N}$. Further, let $\phi(z)=1+B_1z+B_2z^2+\cdots$, where B_n are real with $B_1>0, B_2\geq 0$, and

$$\sigma_1=\frac{[B_2-B_1+(p-\mu)B_1^2]\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2B_1^2(p-\mu)\left(\frac{p+2\delta}{p}\right)^{m}},$$

$$\sigma_2=\frac{[B_2+B_1+(p-\mu)B_1^2]\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2B_1^2(p-\mu)\left(\frac{p+2\delta}{p}\right)^{m}},$$

$$\sigma_3 = \frac{[B_2 + (p-\mu)B_1^2]\gamma_1^2 \left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1^2 (p-\mu)\left(\frac{p+2\delta}{p}\right)^m}.$$

If $f(z) \in A(p)$ belongs to $S^m_{p,\lambda,\mu,\eta,\delta}(\phi)$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \begin{cases} -\dfrac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\upsilon, & \theta \leq \sigma_1, \\ \dfrac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}, & \sigma_1 \leq \theta \leq \sigma_2, \\ \dfrac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\upsilon, & \theta \geq \sigma_2, \end{cases} \tag{3.4.2}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1(p-\mu)\left(\frac{p+2\delta}{p}\right)^m}(1+\upsilon)\left|a_{p+1}\right|^2$$
$$\leq \frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m} \tag{3.4.3}$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2} - \theta a_{p+1}^2\right| + \frac{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1(p-\mu)\left(\frac{p+2\delta}{p}\right)^m}(1-\upsilon)\left|a_{p+1}\right|^2$$
$$\leq \frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m} \tag{3.4.4}$$

where

$$v=\frac{(p-\mu)B_1\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}-\frac{B_2}{B_1}$$

for any complex number θ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right|\leq\frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\max\{1,|v|\} \tag{3.4.5}$$

The results are sharp.

Proof

If $f(z)\in S_{p,\lambda,\mu,\eta,\delta}^m(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots\ \in\Omega$$

such that

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=\phi\big(w(z)\big) \tag{3.4.6}$$

since

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=1+\frac{\gamma_1}{p-\mu}\left(\frac{p+\delta}{p}\right)^m a_{p+1}z$$

$$+\frac{1}{p-\mu}\left[2\gamma_2\left(\frac{p+2\delta}{p}\right)^m a_{p+2}-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}a_{p+1}^2\right]z^2+\cdots$$

we have from (3.4.6),

$$a_{p+1}=\frac{(p-\mu)w_1}{\gamma_1\left(\frac{p+\delta}{p}\right)^m}B_1, \tag{3.4.7}$$

and

$$a_{p+2}=\frac{(p-\mu)}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\{B_1w_2+(B_2+(p-\mu)B_1^2)w_1^2\} \tag{3.4.8}$$

Therefore, we have

$$a_{p+2}-\theta a_{p+1}^{2}=\frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^{m}}\{w_2-vw_1^2\}$$

where

$$v=\frac{(p-\mu)B_1\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^{m}-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}-\frac{B_2}{B_1}$$

The results (3.4.2)-(3.4.4) are established by an application of Lemma 3.1.3 and inequality (3.4.5) by Lemma 3.1.4. To show that the bounds in (3.4.2)-(3.4.4) are sharp, we define the functions $K_{\phi n}$ $(n=2,3,\dots)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}K_{\phi n}(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}K_{\phi n}(z)}=\phi(z^{n-1}),\qquad \left(K_{\phi n}(0)=\left(K_{\phi n}\right)'(0)-1=0\right)$$

and the functions F_r , G_r $(0\le r\le 1)$ defined by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}F_r(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}F_r(z)}=\phi\left(\frac{z(z+r)}{1+rz}\right),\qquad (F_r(0)=F_r'(0)-1=0)$$

and

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}G_r(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}G_r(z)}=\phi\left(-\frac{z(z+r)}{1+rz}\right),\qquad (G_r(0)=G_r'(0)-1=0)$$

respectively, it is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $S_{p,\lambda,\mu,\eta,\delta}^{m}(\phi)$. If $\theta<\sigma_1$ or $\theta>\sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. If $\sigma_1<\theta<\sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta=\sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta=\sigma_2$, then the equality holds if and only if f is G_r or one of its rotations.

Remark 3.4.1

By taking $m = 0$ in Theorem 3.4.2, Theorem 3.3.2 due to Amsheri and Zharkove [8] is obtained.

Remark 3.4.2

By taking $\lambda = \mu = 0$ and $\delta = 0$ in Theorem 3.4.2, Theorem 3.2.2 due to Ail et. al. [2] is obtained.

Remark 3.4.3

By taking $\lambda = \mu = \delta = 0$ and $p = 1$ in Theorem 3.4.2, Corollary 3.2.3 due to Ma and Minda [32] is obtained.

In the similar manner, the coefficient bound for $\left|a_{p+2} - \theta a_{p+1}^2\right|$ of functions in the class $S_{b,p,\lambda,\mu,\eta,\delta}^m(\phi)$ [4] is obtained as follows.

Theorem 3.4.3

Let $0 \leq \theta \leq 1, m \in \mathbb{N}_0, \lambda \geq 0, \mu < p + 1, \eta > \max(\lambda, \mu) - p - 1, \delta \geq 0$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n's are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z) \in A(p)$ belongs to $S_{b,p,\lambda,\mu,\eta,\delta}^m(\phi)$, then for any complex number θ, we have

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \leq \frac{(p-\mu)|b|B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m} \max\{1, |v|\} \tag{3.4.9}$$

where

$$v = \frac{(p-\mu)bB_1\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m - \gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}} - \frac{B_2}{B_1}$$

The result is sharp.

Proof

If $f(z) \in S_{b,p,\lambda,\mu,\eta,\delta}^m(\phi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + \cdots \ \in \Omega$$

such that

$$1+\frac{1}{b}\left(\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-1\right)=\phi\big(w(z)\big) \tag{3.4.10}$$

since

$$1+\frac{1}{b}\left(\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-1\right)=1+\frac{\gamma_1}{b(p-\mu)}\left(\frac{p+\delta}{p}\right)^m a_{p+1}z$$

$$+\frac{1}{b(p-\mu)}\left[2\gamma_2\left(\frac{p+2\delta}{p}\right)^m a_{p+2}-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m} a_{p+1}^2\right]z^2+\cdots$$

we have from (3.4.10),

$$a_{p+1}=\frac{b(p-\mu)w_1}{\gamma_1\left(\frac{p+\delta}{p}\right)^m}B_{1,} \tag{3.4.11}$$

and

$$a_{p+2}=\frac{b(p-\mu)}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\{B_1w_2+(B_2+b(p-\mu)B_1^2)w_1^2\} \tag{3.4.12}$$

Therefore, we have

$$a_{p+2}-\theta a_{p+1}^2=\frac{b(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\{w_2-vw_1^2\}$$

where

$$v=\frac{(p-\mu)bB_1\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}-\frac{B_2}{B_1}.$$

The result (3.4.9) is established by an application of Lemma 3.1.4. To show that the bound in (3.4.9) is sharp, we define the functions $K_{\phi n}$ $(n=2,3,\dots)$ by

$$1+\frac{1}{b}\left(\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}K_{\phi n}(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}K_{\phi n}(z)}-1\right)=\phi(z^{n-1}),$$

where $K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0$. It is clear that the functions $K_{\phi n}$ belong to the class $S^{m}_{b,p,\lambda,\mu,\eta,\delta}(\phi)$, then the equality holds if and only if $f(z)$ is $K_{\phi 2}$ or $K_{\phi 3}$.

Remark 3.4.4

By taking $m = 0$ in Theorem 3.4.3, Theorem 3.3.3 due to Amsheri and Zharkove [8] is obtained.

Remark 3.4.5

By taking $\lambda = \mu = \delta = 0$ and $p = 1$ in Theorem 3.4.3, the corresponding result due to Ravichandran et. al. [49] is obtained.

Chapter 4

Starlikeness and convexity conditions of analytic functions

The main objective of the present chapter is to obtain starlikeness and convexity conditions of analytic and p-valent functions defined in the open unit disk.

4.1 Introduction and preliminaries

In this section, some known results on starlikeness and convexity of analytic and univalent functions are collected. There are many works on the sufficient conditions for starlikeness and convexity of analytic functions, for example [7], [20], [21], [22], [41], [43], [47], [54] and others. In 1975, Silverman [60] proved the following coefficient conditions of functions $f(z) \in A$, that are sufficient for these functions to be in the class $S^*(\alpha)$ or the class $K(\alpha)$ for $0 \leq \alpha < 1$, respectively.

Lemma 4.1.1

Let $0 \leq \alpha < 1$ and the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1-\alpha, \qquad (z \in \mathcal{U}) \qquad (4.1.1)$$

Then $f(z) \in S^*(\alpha)$.

Lemma 4.1.2

Let $0 \leq \alpha < 1$ and the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha, \qquad (z \in \mathcal{U}) \qquad (4.1.2)$$

Then $f(z) \in K(\alpha)$.

The fractional operator has gained much attention by many authors because of its interesting application. Owa and Shen [41] extended the above conditions (4.1.1) and (4.1.2) for functions $f(z) \in A$ involving the fractional derivative operator $\Omega_{0,z}^{\lambda} f(z)$ given by (2.8.3.3) in order to be in the class $S^*(\alpha)$ or the class $K(\alpha)$ for $0 \le \alpha < 1$, as follows.

Lemma 4.1.3

Let $\lambda \ge 0, 0 \le \alpha < 1$ and the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \le \frac{2-\lambda}{2}, \qquad (z \in \mathcal{U}) \qquad (4.1.3)$$

Then $\Omega_{0,z}^{\lambda} f(z) \in S^*(\alpha)$.

Lemma 4.1.4

Let $\lambda \ge 0, 0 \le \alpha < 1$ and the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| \le \frac{2-\lambda}{2}, \qquad (z \in \mathcal{U}) \qquad (4.1.4)$$

Then $\Omega_{0,z}^{\lambda} f(z) \in K(\alpha)$.

Furthermore, by using the fractional derivative operator $P_{0,z}^{\lambda,\mu,\eta} f(z)$ given by (2.8.3.4) for functions $f(z) \in A$, Raina and Nahar [47] generalized the a-fore-mentioned results (4.1.1)-(4.1.4) that deal with starlikeness and convexity as follows.

Lemma 4.1.5

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \ge 0$, $\mu < 2$, $\max(\lambda, \mu) - 2 < \eta \le \lambda\left(1 - \frac{3}{\mu}\right)$ and $0 \le \alpha < 1$. Also, let the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \le \frac{(2-\mu)(2-\lambda+\eta)}{2(2-\mu+\eta)}, \qquad (z \in \mathcal{U}) \qquad (4.1.5)$$

Then $P_{0,z}^{\lambda,\mu,\eta} f(z) \in S^*(\alpha)$.

Lemma 4.1.6

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \geq 0$, $\mu < 2$, $\max(\lambda, \mu) - 2 < \eta \leq \lambda\left(1 - \frac{3}{\mu}\right)$ and $0 \leq \alpha < 1$. Also, let the function $f(z) \in A$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| \leq \frac{(2-\mu)(2-\lambda+\eta)}{2(2-\mu+\eta)}, \qquad (z \in \mathcal{U}) \qquad (4.1.6)$$

Then $P_{0,z}^{\lambda,\mu,\eta} f(z) \in K(\alpha)$.

In 1973, Ruscheweyh and Sheil-Small [54] proved the following important property of analytic functions by using the technique of convolution.

Lemma 4.1.7

Let $\varphi(z)$ and $g(z)$ be analytic functions in $\mathcal{U}$ and satisfy $\varphi(0) = g(0) = 0$, $\varphi'(0) \neq 0$, $g'(0) \neq 0$. Also, let

$$\varphi(z) * \left\{\frac{1+abz}{1-bz} g(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\}) \qquad (4.1.7)$$

for a and b on the unit circle. Then for a function $F(z)$ analytic in $\mathcal{U}$ such that $\operatorname{Re}\{F(z)\} > 0$ satisfies the inequality

$$\operatorname{Re}\left\{\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right\} > 0, \qquad (z \in \mathcal{U}) \qquad (4.1.8)$$

Many known results on starlikeness and convexity were obtained by using Lemma 4.1.7. For example, Owa and Shen [41] proved the following results for the fractional derivative operator $\Omega_{0,z}^{\lambda} f(z)$ of functions $f(z) \in A$.

Lemma 4.1.8

Let $\lambda < 1$, $0 \leq \alpha < 1$ and the function $f(z) \in A$ be in the class $S^*(\alpha)$. If $f(z)$ satisfies

$$h(z) * \left\{\frac{1+abz}{1-bz} f(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle, where

$$h(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n \tag{4.1.9}$$

Then $\Omega_{0,z}^{\lambda}f(z)\in S^*(\alpha)$.

Lemma 4.1.9

Let $\lambda<1,\ 0\le\alpha<1$ and the function $f(z)\in A$ be in the class $K(\alpha)$. If $f(z)$ satisfies

$$h(z)*\left\{\frac{1+abz}{1-bz}zf'(z)\right\}\neq 0,\qquad (z\in\mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle and $h(z)$ is given by (4.1.9). Then $\Omega_{0,z}^{\lambda}f(z)\in K(\alpha)$.

Also, Raina and Nahar [47] used Lemma 4.1.7 to prove the following results which deal with starlikeness and convexity conditions for the fractional derivative operator $P_{0,z}^{\lambda,\mu,\eta}f(z)$ of functions $f(z)\in A$.

Lemma 4.1.10

Let $\lambda,\ \mu,\ \eta\in\ \mathbb{R}$ such that $\lambda\ge 0,\ \mu<2,\ \max(\lambda,\mu)-2<\eta\le\lambda\left(1-\frac{3}{\mu}\right)$ and $0\le\alpha<1$. Also, let the function $f(z)\in A$ be in the class $S^*(\alpha)$. If $f(z)$ satisfies

$$h(z)*\left\{\frac{1+abz}{1-bz}f(z)\right\}\neq 0,\qquad (z\in\mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle, where

$$h(z)=z+\sum_{n=2}^{\infty}\frac{(2)_{n-1}(2-\mu+\eta)_{n-1}}{(2-\mu)_{n-1}(2-\lambda+\eta)_{n-1}}z^n \tag{4.1.10}$$

Then $P_{0,z}^{\lambda,\mu,\eta}f(z)\in S^*(\alpha)$.

Lemma 4.1.11

Let λ, μ, $\eta\in\mathbb{R}$ such that $\lambda\ge 0$, $\mu<2$, $\max(\lambda,\mu)-2<\eta\le\lambda\left(1-\frac{3}{\mu}\right)$ and $0\le\alpha<1$. Also, let the function $f(z)\in A$ be in the class $K(\alpha)$. If $f(z)$ satisfies

$$h(z) * \left\{\frac{1+abz}{1-bz} zf'(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle and $h(z)$ is given by (4.1.10). Then $P_{0,z}^{\lambda,\mu,\eta} f(z) \in K(\alpha)$.

Next, the following results due to Jack [23] and Nunokawa [38] (Lemma 4.1.12 and Lemma 4.1.13 below) which are popularly known as Jack's Lemma and Nonokawa's Lemma, respectively in the literature have been applied in proving many results on starlikeness and convexity of analytic functions.

Lemma 4.1.12

Let $w(z)$ be non-constant and analytic function in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r, 0 < r < 1$ at the point z_0, then $z_0 w'(z_0) = cw(z_0)$, where $c \geq 1$.

Lemma 4.1.13

Let $p(z)$ be an analytic function in $\mathcal{U}$ with $p(0) = 1$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\text{Re}\{p(z)\} > 0 \quad (|z| < |z_0|), \quad \text{Re}\{p(z_0)\} = 0, \;\; p(z_0) \neq 0$$

then

$$p(z_0) = i\,a, \qquad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a + \frac{1}{a}\right)$$

where $a \neq 0$ and $c \geq 1$.

By making use of these results, Irmak and Piejko [21] investigated the following conditions for starlikeness and convexity of functions $f(z) \in A_n$.

Lemma 4.1.14

Let $z \in \mathcal{U}, 0 \leq \alpha < 1$ and the function $f(z) \in A_n$.

1. If

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \frac{-(3+\alpha)}{2(1+\alpha)} \tag{4.1.11}$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1+\alpha}{2} \tag{4.1.12}$$

2. If

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -1 \tag{4.1.13}$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{4.1.14}$$

Lemma 4.1.15

Let $z \in \mathcal{U}, 0 \leq \alpha < 1$ and the function $f(z) \in A_n$.

1. If

$$\mathrm{Re}\left\{\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)}\right\} > \frac{\alpha - 1}{2(1+\alpha)} \tag{4.1.15}$$

then

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1+\alpha}{2}\,, \tag{4.1.16}$$

2. If

$$\mathrm{Re}\left\{\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)} - \frac{zf''(z)}{f'(z)}\right\} > 0 \tag{4.1.17}$$

then

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{4.1.18}$$

In this chapter, motivated by a-fore-mentioned works, some known starlikeness and convexity conditions of p-valent functions are studied. Moreover, various new results that deal with the starlikness and convexity of p-valent functions associated with generalized linear operator are also obtained.

4.2 Conditions for p-valent functions

Motivated by the work of Silverman [60], Owa [40] proved the following sufficient coefficient conditions for functions $f(z) \in A(p)$ to be in the class $S^*(p,\alpha)$ or the class $K(p,\alpha)$ for $0 \leq \alpha < p$ and $p \in \mathbb{N}$.

Theorem 4.2.1

Let $0 \leq \alpha < p, p \in \mathbb{N}$ and the function $f(z) \in A(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n-\alpha)\left|a_{p+n}\right| \leq p-\alpha, \qquad (z \in \mathcal{U}) \qquad (4.2.1)$$

Then $f(z) \in S^*(p,\alpha)$.

Theorem 4.2.2

Let $0 \leq \alpha < p, p \in \mathbb{N}$ and the function $f(z) \in A(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)\left|a_{p+n}\right| \leq p(p-\alpha), \qquad (z \in \mathcal{U}) \qquad (4.2.2)$$

Then $f(z) \in K(p,\alpha)$.

Next, by using the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p}f(z)$ given by (2.8.3.5) of functions $f(z) \in A(p)$, Amsheri and Zharkova [7] obtained the sufficient conditions for starlikeness and convexity, which generalize the works by [40], [41],[47] and [60] as follows.

Theorem 4.2.3

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \geq 0$, $\mu < p+1$, $\max(\lambda,\mu) - p - 1 < \eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty}\frac{p+n-\alpha}{p-\alpha}\left|a_{p+n}\right| \leq \frac{(p+1-\mu)(p+1+\eta-\lambda)}{(p+1)(p+1+\eta-\mu)}, \qquad (z \in \mathcal{U})$$

$$(4.2.3)$$

Then $M_{0,z}^{\lambda,\mu,\eta,p}f(z) \in S^*(p,\alpha)$.

Proof

We have from (2.8.3.5)

$$M_{0,z}^{\lambda,\mu,\eta,p} f(z) = z^p + \sum_{n=1}^{\infty} \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n}$$

we observe that the function $\gamma_n(\lambda,\mu,\eta,p)$ defined by (2.8.3.6) satisfies the inequality $\gamma_n(\lambda,\mu,\eta,p) \geq \gamma_{n+1}(\lambda,\mu,\eta,p)$, $\forall n \in \mathbb{N}$, provided that $\eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$. Thereby, showing that $\gamma_n(\lambda,\mu,\eta,p)$ is non-increasing. Thus under the hypothesis of the theorem, we have

$$\gamma_n(\lambda,\mu,\eta,p) \leq \gamma_1(\lambda,\mu,\eta,p) = \frac{(p+1)(p+1+\eta-\mu)}{(p+1-\mu)(p+1+\eta-\lambda)} \qquad (4.2.4)$$

Therefore, (4.2.3) and (4.2.4) yields

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \gamma_n(\lambda,\mu,\eta,p) |a_{p+n}|$$

$$\leq \gamma_1(\lambda,\mu,\eta,p) \sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} |a_{p+n}| \leq 1$$

Hence, by Lemma 4.2.1, we conclude that $M_{0,z}^{\lambda,\mu,\eta,p} f(z) \in S^*(p,\alpha)$.

Theorem 4.2.4

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \geq 0$, $\mu < p+1$, $\max(\lambda,\mu) - p - 1 < \eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$. If $f(z)$ satisfies

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n-\alpha)}{p(p-\alpha)} |a_{p+n}| \leq \frac{(p+1-\mu)(p+1+\eta-\lambda)}{(p+1)(p+1+\eta-\mu)}, \quad (z \in \mathcal{U}) \qquad (4.2.5)$$

Then $M_{0,z}^{\lambda,\mu,\eta,p} f(z) \in K(p,\alpha)$.

Proof

From (4.2.5) and (4.2.4), we have

$$\sum_{n=1}^{\infty}\frac{(p+n)(p+n-\alpha)}{p(p-\alpha)}\gamma_n(\lambda,\mu,\eta,p)\left|a_{p+n}\right|$$

$$\leq \gamma_1(\lambda,\mu,\eta,p)\sum_{n=1}^{\infty}\frac{(p+n)(p+n-\alpha)}{p(p-\alpha)}\left|a_{p+n}\right| \leq 1$$

Hence, by Lemma 4.2.2, we conclude that $M_{0,z}^{\lambda,\mu,\eta,p}f(z) \in K(p,\alpha)$.

Next, by applying Lemma 4.1.7, starlikeness and convexity conditions for functions $f(z) \in A(p)$ involving the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p}f(z)$ in terms of convolution were obtained by Amsheri and Zharkova [7] as follows.

Theorem 4.2.5

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \geq 0, \mu < p+1, \max(\lambda,\mu) - p - 1 < \eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$ be in the class $S^*(p,\alpha)$. If $f(z)$ satisfies

$$\Psi(z) * \left\{\frac{1+abz}{1-bz}f(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle, where

$$\Psi(z) = z^p + \sum_{n=1}^{\infty}\frac{(p+1)_n(p+1+\eta-\mu)_n}{(p+1-\mu)_n(p+1+\eta-\lambda)_n}z^{p+n} \qquad (4.2.6)$$

Then $M_{0,z}^{\lambda,\mu,\eta,p}f(z) \in S^*(p,\alpha)$.

Proof

Using (2.8.3.5) and (4.2.6), we have

$$M_{0,z}^{\lambda,\mu,\eta,p}f(z) = z^p + \sum_{n=1}^{\infty}\gamma_n(\lambda,\mu,\eta,p)a_{p+n}z^{p+n}$$

$$= (\Psi * f)(z) \qquad (4.2.7)$$

By setting $\varphi(z) = \Psi(z)$, $g(z) = f(z)$ and $F(z) = \frac{zf'(z)}{f(z)} - \alpha$ in Lemma 4.1.7, we find with the help of (4.2.7) that

$$\operatorname{Re}\left\{\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right\} > 0$$

$$\Rightarrow \operatorname{Re}\left\{\frac{(\Psi * zf')(z)}{(\Psi * f)(z)}\right\} - \alpha > 0$$

$$\Rightarrow \operatorname{Re}\left\{\frac{z(\Psi * f)'(z)}{(\Psi * f)(z)}\right\} - \alpha > 0$$

$$\Rightarrow \operatorname{Re}\left\{\frac{z\left(M_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta,p} f(z)}\right\} - \alpha > 0$$

$$\Rightarrow M_{0,z}^{\lambda,\mu,\eta,p} f(z) \in S^*(p,\alpha)$$

and the proof is complete.

Theorem 4.2.6

Let λ, μ, $\eta \in \mathbb{R}$ such that $\lambda \geq 0, \mu < p+1, \max(\lambda,\mu) - p - 1 < \eta \leq \lambda\left(1 - \frac{p+2}{\mu}\right)$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$ be in the class $K(p,\alpha)$. If $f(z)$ satisfies

$$\Psi(z) * \left\{\frac{1 + abz}{1 - bz} zf'(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle and $\Psi(z)$ is given by (4.2.6). Then $M_{0,z}^{\lambda,\mu,\eta,p} f(z) \in K(p,\alpha)$.

Proof

Using (2.8.3.5) and Theorem 4.2.5, we observe that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha)$$

$$\Rightarrow M_{0,z}^{\lambda,\mu,\eta,p}\left(\frac{zf'(z)}{p}\right) \in S^*(p,\alpha)$$

$$\Leftrightarrow \left(\Psi * \frac{zf'}{p}\right)(z) \in S^*(p,\alpha)$$

$$\Leftrightarrow \frac{z(\Psi * f)'(z)}{p} \in S^*(p,\alpha)$$

$$\Leftrightarrow (\Psi * f)(z) \in K(p,\alpha)$$

$$\Leftrightarrow M_{0,z}^{\lambda,\mu,\eta,p} f(z) \in K(p,\alpha)$$

and the proof is complete.

4.3 Conditions for p-valent functions associated with generalized differential operator

Motivated by a-fore-mentioned works in the current chapter, various new sufficient conditions for starlikeness and convexity of functions $f(z) \in A(p)$ associated with generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ given by (2.8.3.7) are investigated according to Amsheri and Abouthfeerah [5].

The first characterization property for starlikeness of the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ is given as follows.

Theorem 4.3.1

Let $m \in \mathbb{N}_0$, $\lambda \geq 0, \mu < p+1, \eta > \max(\lambda,\mu) - p - 1, \delta \geq 0$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ defined by (2.8.3.7). If $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left|a_{p+n}\right| \leq p-\alpha \tag{4.3.1}$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). Then $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in S^*(p,\alpha)$.

Proof

It is sufficient to show that the values of $\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}$ lie in a circle centered at $w=p$ whose radius is $(p-\alpha)$. Then we obtain

$$\left|\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-p\right|=\left|\frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)na_{p+n}z^{p+n}}{z^p+\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)a_{p+n}z^{p+n}}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)n|a_{p+n}|\,|z|^n}{1-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)|a_{p+n}||z|^n}$$

$$\leq\frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)n|a_{p+n}|}{1-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)|a_{p+n}|}.$$

The last expression is bounded above by $(p-\alpha)$, if

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)n|a_{p+n}|$$

$$\leq(p-\alpha)\left[1-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}\gamma_n(\lambda,\mu,\eta,p)|a_{p+n}|\right].$$

But the inequality is equivalent to (4.3.1) and true by hypothesis. Hence, $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\in S^*(p,\alpha)$.

In the similar manner, the characterization property for convexity of the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ can be proved as follows.

Theorem 4.3.2

Let $m\in\mathbb{N}_0$, $\lambda\geq 0,\mu<p+1,\eta>\max(\lambda,\mu)-p-1$, $\delta\geq 0$, $0\leq\alpha<p$ and $p\in\mathbb{N}$. Also, let the function $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ defined by (2.8.3.7). If $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ satisfies

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)|a_{p+n}| \leq p(p-\alpha) \tag{4.3.2}$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6).Then $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z) \in K(p,\alpha)$.

Proof

It is sufficient to show that the values of $\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)''}{\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}$ lie in a circle centered at $w = p-1$ whose radius is $(p-\alpha)$. Then we obtain

$$\left|\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)''}{\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}+1-p\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)n(p+n)a_{p+n}z^{p+n-1}}{pz^{p-1}+\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)a_{p+n}z^{p+n-1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)n(p+n)|a_{p+n}|\,|z|^n}{p-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)|a_{p+n}||z|^n}$$

$$\leq \frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)n(p+n)|a_{p+n}|}{p-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)|a_{p+n}|}.$$

The last expression is bounded above by $(p-\alpha)$, if

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)n(p+n)|a_{p+n}|$$

$$\leq (p-\alpha)\left[p-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)|a_{p+n}|\right].$$

But the inequality is equivalent to (4.3.2) and true by hypothesis. Hence, we conclude that $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in K(p,\alpha)$.

Remark 4.3.1

1. By setting $p = 1$ and $\lambda = \mu = \delta = 0$ in Theorem 4.3.1 and Theorem 4.3.2, respectively, Lemma 4.1.1 and Lemma 4.1.2 due to Silverman [60] are obtained.
2. By setting $\lambda = \mu = \delta = 0$ in Theorem 4.3.1 and Theorem 4.3.2, respectively, Theorem 4.2.1 and Theorem 4.2.2 due to Owa [40] are obtained.

Now, by making use of Lemma 4.1.7, the following starlikeness and convexity results in terms of convolution are established.

Theorem 4.3.3

Let $m \in \mathbb{N}_0, \lambda \geq 0, \mu < p+1, \eta > \max(\lambda,\mu) - p - 1, \delta \geq 0, 0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$ be in the class $S^*(p,\alpha)$. If $f(z)$ satisfies

$$\psi(z) * \left\{\frac{1+abz}{1-bz} f(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\})$$

for each a and b on the unit circle, where

$$\psi(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p) z^{p+n} \tag{4.3.3}$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). Then $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in S^*(p,\alpha)$.

Proof

Using (2.8.3.7) and (4.3.3), we have

$$\begin{aligned} N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \\ &= (\psi * f)(z) \end{aligned} \tag{4.3.4}$$

By setting $\varphi(z) = \psi(z)$, $g(z) = f(z)$ and $F(z) = \frac{zf'(z)}{f(z)} - \alpha$ in Lemma 4.1.7, we find with the help of (4.3.4) that

$$\mathrm{Re}\left\{\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right\} > 0$$

$$\Rightarrow \mathrm{Re}\left\{\frac{(\psi * zf')(z)}{(\psi * f)(z)}\right\} - \alpha > 0$$

$$\Rightarrow \mathrm{Re}\left\{\frac{z(\psi * f)'(z)}{(\psi * f)(z)}\right\} - \alpha > 0$$

$$\Rightarrow \mathrm{Re}\left\{\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)}\right\} - \alpha > 0$$

$$\Rightarrow N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in S^*(p,\alpha)$$

and the proof is complete.

Theorem 4.3.4

Let $m \in \mathbb{N}_0$, $\lambda \geq 0, \mu < p+1, \eta > \max(\lambda,\mu) - p - 1$, $\delta \geq 0$, $0 \leq \alpha < p$ and $p \in \mathbb{N}$. Also, let the function $f(z) \in A(p)$ be in the class $K(p,\alpha)$. If $f(z)$ satisfies

$$\psi(z) * \left\{\frac{1+abz}{1-bz} zf'(z)\right\} \neq 0, \qquad (z \in \mathcal{U}\backslash\{0\}) \qquad (4.3.5)$$

for each a and b on the unit circle and $\psi(z)$ is given by (4.3.3). Then $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in K(p,\alpha)$.

Proof

Using (2.8.3.7) and Theorem 4.3.3, we observe that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha)$$

$$\Rightarrow N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right) \in S^*(p,\alpha)$$

$$\Leftrightarrow \left(\psi * \frac{zf'}{p}\right)(\mathrm{z}) \in S^*(p,\alpha)$$

$$\Leftrightarrow \frac{z(\psi * f)'(\mathrm{z})}{p} \in S^*(p,\alpha)$$

$$\Leftrightarrow (\psi * f)(\mathrm{z}) \in K(p,\alpha)$$

$$\Leftrightarrow N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \in K(p,\alpha)$$

and the proof is complete.

Remark 4.3.2

1. Letting $p = 1,\ \delta = 0$ and $\lambda =\ \mu$ in Theorem 4.3.3 and Theorem 4.3.4, respectively, Lemma 4.1.8 and Lemma 4.1.9 due to Owa and Shen [41] are obtained.
2. Letting $p = 1$ and $\delta = 0$ in Theorem 4.3.3 and Theorem 4.3.4, respectively, Lemma 4.1.10 and Lemma 4.1.11 due to Raina and Nahar [47] are obtained.
3. Letting $m = 0$ in Theorem 4.3.3 and Theorem 4.3.4, respectively, Theorem 4.2.5 and Theorem 4.2.6 due to Amsheri and Zharkova [7] are obtained.

Next, by applying Lemma 4.1.12 and Lemma 4.1.13, other starlikeness and convexity conditions for the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ are investigated.

Theorem 4.3.5

Let $z \in \mathcal{U},\ m \in \mathbb{N}_0,\ \lambda \geq 0, \mu < p + 1,\ \eta > \max(\lambda, \mu) - p - 1, \delta \geq 0, 0 \leq \alpha < 1, p \in \mathbb{N}$ and $f(z) \in A(p)$.

1. If

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}\right\}$$

$$>\frac{-(3+\alpha)}{2(1+\alpha)} \qquad (4.3.6)$$

then

$$\operatorname{Re}\left\{\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}\right\}>\frac{1+\alpha}{2} \qquad (4.3.7)$$

2. If

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}\right\}$$

$$>-1 \qquad (4.3.8)$$

then

$$\operatorname{Re}\left\{\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}\right\}>\alpha \qquad (4.3.9)$$

Proof

First, we prove (1). Since

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=1+d_1z+d_2z^2+\cdots, \qquad (z\in\mathcal{U})$$

Define the function $w(z)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=\frac{1+\alpha w(z)}{1+w(z)} \qquad (4.3.10)$$

for $0\leq\alpha<1$ and $z\in\mathcal{U}$. It is clear that $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. Also, we can find from (4.3.10) that

$$\frac{z\left(N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}$$

$$=\frac{\alpha zw'(z)}{1+\alpha w(z)}-\frac{zw'(z)}{1+w(z)} \qquad (4.3.11)$$

by using (2.8.3.9) to (4.3.11), we have

$$(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}$$

$$=\frac{\alpha zw'(z)}{1+\alpha w(z)}-\frac{zw'(z)}{1+w(z)}-1 \qquad (4.3.12)$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$$

then by Lemma 4.1.12, we have

$$z_0w'(z_0)=cw(z_0), \qquad (c\ge 1)$$

Therefore, since $w(z_0)=e^{i\theta}$, we obtain

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z_0)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z_0)}-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z_0)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z_0)}\right\}$$

$$=\operatorname{Re}\left\{\frac{\alpha z_0w'(z_0)}{1+\alpha w(z_0)}-\frac{z_0w'(z_0)}{1+w(z_0)}-1\right\}$$

$$=\operatorname{Re}\left\{\frac{\alpha ce^{i\theta}}{1+\alpha e^{i\theta}}-\frac{ce^{i\theta}}{1+e^{i\theta}}-1\right\}\le\frac{-(3+\alpha)}{2(1+\alpha)}$$

which is a contradiction to the condition (4.3.6). Therefore, $|w(z)|<1$ for all $z\in\mathcal{U}$. Hence (4.3.10) yields

$$\left|\frac{1-\dfrac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}}{\dfrac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-\alpha}\right|=|w(z)|<1, \quad (0\le\alpha<1;\ z\in\mathcal{U})$$

which implies the inequality (4.3.7). This completes the proof of (1) in the Theorem 4.3.5.

For the proof of (2), we define a new function $p(z)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=\alpha+(1-\alpha)p(z) \tag{4.3.13}$$

where $0\le\alpha<1, z\in\mathcal{U}$ and $p(z)$ is analytic in $\mathcal{U}$ with $p(0)=1$. Then we find from (4.3.13) that

$$\frac{z\left(N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}$$
$$=\frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \tag{4.3.14}$$

by using (2.8.3.9) to (4.3.14), we have

$$(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}+1$$
$$=\frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \tag{4.3.15}$$

If there exists a point $z_0\in\mathcal{U}$ such that

$$\mathrm{Re}\{p(z)\}>0 \quad (|z|<|z_0|),$$
$$\mathrm{Re}\{p(z_0)\}=0,\ \ p(z_0)\neq 0, \qquad (z\in\mathcal{U})$$

Then by using Lemma 4.1.13, we have

$$p(z_0)=i\,a,\quad \frac{z_0p'(z_0)}{p(z_0)}=i\frac{c}{2}\left(a+\frac{1}{a}\right), \qquad (a\neq 0; c\ge 1)$$

Thus from (4.3.15), we have

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}f(z_0)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z_0)}\right.$$

$$\left.-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z_0)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z_0)}+1\right\}$$

$$=\operatorname{Re}\left\{\frac{(1-\alpha)z_0p'(z_0)}{p(z_0)}\frac{p(z_0)}{\alpha+(1-\alpha)p(z_0)}\right\}$$

$$=\frac{-c\alpha(1-\alpha)(1+a^2)}{2[\alpha^2+a^2(1-\alpha)^2]}\leq 0, \qquad (0\leq\alpha<1)$$

which contradicts the condition (4.3.8). Hence $\operatorname{Re}\{p(z)\}>0$ for all $z\in\mathcal{U}$ and the equality (4.3.13) implies the condition (4.3.9). Therefore, the proof of the Theorem 4.3.5 is completed.

Now, in order to obtain the sufficient condition for convexity of the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$, we put $\frac{zf'(z)}{p}$ instead of $f(z)$ in the Theorem 4.3.5.

Theorem 4.3.6

Let $z\in\mathcal{U}$, $m\in\mathbb{N}_0$, $\lambda\geq 0, \mu<p+1$, $\eta>\max(\lambda,\mu)-p-1, \delta\geq 0, 0\leq\alpha<1, p\in\mathbb{N}$ and $f(z)\in A(p)$.

1. If

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right.$$

$$\left.-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right\}>\frac{-(3+\alpha)}{2(1+\alpha)} \qquad (4.3.16)$$

then

$$\operatorname{Re}\left\{\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right\}>\frac{1+\alpha}{2} \qquad (4.3.17)$$

2. If

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right.$$

$$\left.-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right\}>-1 \tag{4.3.18}$$

then

$$\operatorname{Re}\left\{\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}\right\}>\alpha \tag{4.3.19}$$

Proof

First, we prove (1). Since

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}=1+d_1\,z+d_2\,z^2+\cdots,\qquad (z\in\mathcal{U})$$

Define the function $w(z)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}=\frac{1+\alpha w(z)}{1+w(z)} \tag{4.3.20}$$

for $0\le\alpha<1$ and $z\in\mathcal{U}$. It is clear that $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. Also, we can find from (4.3.20) that

$$\frac{z\left(N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)\right)'}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}-\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}$$

$$= \frac{\alpha z w'(z)}{1+\alpha w(z)} - \frac{z w'(z)}{1+w(z)} \tag{4.3.21}$$

by using (2.8.3.9) to (4.3.21), we have

$$(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)} -$$

$$(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{zf'(z)}{p}\right)}$$

$$= \frac{\alpha z w'(z)}{1+\alpha w(z)} - \frac{z w'(z)}{1+w(z)} - 1 \tag{4.3.22}$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$$

then by Lemma 4.1.12, we have

$$z_0 w'(z_0) = c w(z_0), \qquad (c \geq 1)$$

Therefore, since $w(z_0) = e^{i\theta}$, we obtain

$$\mathrm{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}\right.$$

$$\left. -(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}\right\}$$

$$= \mathrm{Re}\left\{\frac{\alpha z_0 w'(z_0)}{1+\alpha w(z_0)} - \frac{z_0 w'(z_0)}{1+w(z_0)} - 1\right\}$$

$$= \mathrm{Re}\left\{\frac{\alpha c e^{i\theta}}{1+\alpha e^{i\theta}} - \frac{c e^{i\theta}}{1+e^{i\theta}} - 1\right\} \leq \frac{-(3+\alpha)}{2(1+\alpha)}$$

which is a contradiction to the condition (4.3.16). Therefore, $|w(z)| < 1$ for all $z \in \mathcal{U}$. Hence (4.3.20) yields

$$\left| \frac{1 - \dfrac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}}{\dfrac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)} - \alpha} \right| = |w(z)| < 1, \quad (0 \le \alpha < 1; z \in \mathcal{U})$$

which implies the inequality (4.3.17). This completes the proof of (1) in the Theorem.

For the proof of (2), we define a new function $p(z)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)} = \alpha + (1-\alpha)p(z) \tag{4.3.23}$$

where $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1, 0 \le \alpha < 1$ and $z \in \mathcal{U}$. Then we find from (4.3.23) that

$$\frac{z\left(N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)\right)'}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)} - \frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)\right)'}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}$$

$$= \frac{(1-\alpha)zp'(z)}{\alpha + (1-\alpha)p(z)} \tag{4.3.24}$$

by using (2.8.3.9) to (4.3.24), we have

$$(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)} -$$

$$(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\dfrac{zf'(z)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\dfrac{zf'(z)}{p}\right)} + 1$$

$$= \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \tag{4.3.25}$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re}\{p(z)\} > 0 \quad (|z| < |z_0|),$$
$$\operatorname{Re}\{p(z_0)\} = 0, \;\; p(z_0) \neq 0, \qquad\qquad (\, z \in \mathcal{U}\,)$$

Then by using Lemma 4.1.13, we have

$$p(z_0) = i\,a, \qquad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a+\frac{1}{a}\right), \qquad\qquad (a \neq 0, c \geq 1)$$

Thus from (4.3.25), we have

$$\operatorname{Re}\left\{(p-\mu-1)\frac{N_{0,z}^{m,\lambda+2,\mu+2,\eta+2,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}\right.$$
$$\left.-(p-\mu)\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}\left(\frac{z_0 f'(z_0)}{p}\right)}+1\right\}$$
$$= \operatorname{Re}\left\{\frac{(1-\alpha)z_0 p'(z_0)}{p(z_0)}\frac{p(z_0)}{\alpha+(1-\alpha)p(z_0)}\right\}$$
$$= \frac{-c\alpha(1-\alpha)(1+a^2)}{2[\alpha^2+a^2(1-\alpha)^2]} \leq 0$$

which contradicts the condition (4.3.18). Hence $\operatorname{Re}\{p(z)\} > 0$ for all $z \in \mathcal{U}$ and the equality (4.3.23) implies the condition (4.3.19). Therefore, the proof of the Theorem 4.3.6 is completed.

Remark 4.3.3

By setting $\lambda = \mu = \delta = 0$ in Theorem 4.3.5, the sufficient conditions for starlikeness of p-valent functions in $\mathcal{U}$ is obtained as follows.

Corollary 4.3.7

Let $z \in \mathcal{U},\; 0 \leq \alpha < 1$ and $f(z) \in A(p)$.

1. If

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}>\frac{-(3+\alpha)}{2(1+\alpha)} \tag{4.3.26}$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{pf(z)}\right\}>\frac{1+\alpha}{2}\,, \tag{4.3.27}$$

2. If

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}>-1 \tag{4.3.28}$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{pf(z)}\right\}>\alpha \tag{4.3.29}$$

Remark 4.3.4

By setting $\lambda=\mu=\delta=0$ in Theorem 4.3.6, the sufficient conditions for convexity of p-valent functions in $\mathcal{U}$ is obtained as follows.

Corollary 4.3.8

Let $z\in\mathcal{U},\ 0\le\alpha<1$ and $f(z)\in A(p)$.

1. If

$$\mathrm{Re}\left\{\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right\}>\frac{\alpha-1}{2(1+\alpha)} \tag{4.3.30}$$

then

$$\mathrm{Re}\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\frac{1+\alpha}{2} \tag{4.3.31}$$

2. If

$$\mathrm{Re}\left\{\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right\}>0 \tag{4.3.32}$$

then

$$\mathrm{Re}\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha \tag{4.3.33}$$

Chapter 5

Certain classes of analytic functions with negative coefficients

This chapter is devoted to study certain classes of analytic and p-valent functions whose non-zero coefficients, from the second on, are negative with an aim to obtain some properties.

5.1 Introduction and preliminaries

For univalent functions, the well-known classes $T^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$ with negative coefficients, which obtained by taking intersection, respectively, of the classes $S^*(\alpha)$ and $K(\alpha)$ with T, that are,

$$T^*(\alpha) = S^*(\alpha) \cap T, C(\alpha) = K(\alpha) \cap T$$

These classes were introduced and studied by Silverman [60]. Results concerning coefficient inequalities, distortion, covering theorems, order of starlikeness, radius of convexity theorems and extreme points are obtained by author. Further, Owa [40] introduced the classes $T^*(p, \alpha)$ and $C(p, \alpha)$ of p-valent starlike and convex functions of order $\alpha, 0 \leq \alpha < p$ with negative coefficients which are extensions of the familiar classes $T^*(\alpha)$ and $C(\alpha)$, respectively, when $p = 1$, that are

$$T^*(1, \alpha) = T^*(\alpha), C(1, \alpha) = C(\alpha)$$

Many authors have defined various classes of univalent and p-valent functions with negative coefficients and studied their geometric and analytic properties, such as [10], [12], [40], [42], [56], [59], [60], [61], [63] and others.

In this chapter, motivated essentially by a-fore-mentioned works, in order to solve many problems such as, coefficient bounds, distortion properties, Hadamard product (convolution) properties, closure properties, extreme points, radius of close-to-convexity, radius of starlikeness, radius of convexity, class-preserving integral operators and integral means inequalities, the classes $T^*(p,\alpha)$ and $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ of analytic and p-valent functions with negative coefficients are defined and studied.

Now, in order to prove the results concerning integral means inequality, the following lemma due to Littlewood [28] is needed.

Lemma 5.1

If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $g(z) \prec f(z)$, then for $\tau > 0$ and $0 < r < 1$, we have

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^\tau d\theta \le \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\tau d\theta, \qquad \left(z = re^{i\theta}\right) \qquad (5.1.1)$$

5.2 On a class of p-valent functions

In this section, various properties for functions belonging to the class $T^*(p,\alpha)$ according to Owa [40] and Sălăgean et. al. [56] are studied. The class $T^*(p,\alpha)$ is defined as follows.

Definition 5.2.1 [40]

A function $f(z) \in T(p)$ is said to be p-valent starlike function of order α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathcal{U}) \qquad (5.2.1)$$

for $0 \le \alpha < p$ and $p \in \mathbb{N}$. The class of all p-valent starlike functions of order α with negative coefficients is denoted by $T^*(p,\alpha)$.

Notice that, for $p = 1$, the class $T^*(p,\alpha)$ reduces to the class $T^*(\alpha)$ which was introduced by Silverman [60].

5.2.1 Coefficient bounds

The sufficient and necessary conditions for functions to be in the class $T^*(p,\alpha)$ according to Owa [40] can be obtained as follows.

Theorem 5.2.1.1

Let the function $f(z)$ be defined by (2.2.2). Then $f(z)$ belongs to the class $T^*(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}(p+n-\alpha)a_{p+n} \le p-\alpha \tag{5.2.1.1}$$

The result is sharp.

Proof

Assume that the inequality (5.2.1.1) holds. Then we obtain

$$\begin{aligned}\left|\frac{z\,f'(z)}{f(z)}-p\right| &= \left|\frac{-\sum_{n=1}^{\infty} na_{p+n}\,z^{p+n}}{z^p-\sum_{n=1}^{\infty} a_{p+n}\,z^{p+n}}\right| \\ &\le \frac{\sum_{n=1}^{\infty} na_{p+n}|z|^n}{1-\sum_{n=1}^{\infty} a_{p+n}\,|z|^n} \\ &\le \frac{\sum_{n=1}^{\infty} na_{p+n}}{1-\sum_{n=1}^{\infty} a_{p+n}} \\ &\le p-\alpha\end{aligned}$$

This shows that the values of $z\,f'(z)/f(z)$ lie in a circle centered at $w=p$ whose radius is $(p-\alpha)$. Hence $f(z)\in T^*(p,\alpha)$. Conversely, suppose that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{pz^p-\sum_{n=1}^{\infty}(p+n)a_{p+n}\,z^{p+n}}{z^p-\sum_{n=1}^{\infty} a_{p+n}\,z^{p+n}}\right\} > \alpha$$

for $0\le\alpha<p$, $p\in\mathbb{N}$ and $z\in\mathcal{U}$. Choosing values of z on the real axis so that $zf'(z)/f(z)$ is real, and letting $z\to 1^-$ through real axis, we can see that

$$p-\sum_{n=1}^{\infty}(p+n)a_{p+n} \ge \alpha\left(1-\sum_{n=1}^{\infty} a_{p+n}\right)$$

Thus we have the required inequality (5.2.1.1). Finally, the function $f(z)$ given by

$$f(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n}, \qquad (p, n \in \mathbb{N}) \qquad (5.2.1.2)$$

is an extremal function for the Theorem 5.2.1.1.

Corollary 5.2.1.2

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Then

$$a_{p+n} \leq \frac{p-\alpha}{p+n-\alpha}, \qquad (p, n \in \mathbb{N}) \qquad (5.2.1.3)$$

Equality is attained for the function $f(z)$ given by (5.2.1.2).

Remark 5.2.1.1

Setting $p = 1$ in Theorem 5.2.1.1, the corresponding result proved by Silverman [60] is obtained as follows.

Corollary 5.2.1.3

Let the function $f(z)$ be defined by (2.1.3). Then $f(z)$ belongs to the class $T^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} (n-\alpha) a_n \leq 1-\alpha \qquad (5.2.1.4)$$

The result is sharp.

Remark 5.2.1.2

Setting $p = 1$ in Corollary 5.2.1.2, the corresponding result proved by Silverman [60] is obtained as follows.

Corollary 5.2.1.4

Let the function $f(z)$ defined by (2.1.3) be in the class $T^*(\alpha)$. Then

$$a_n \leq \frac{1-\alpha}{n-\alpha}, \qquad (n \geq 2) \qquad (5.2.1.5)$$

The result is sharp for the function defined by

$$f(z) = \mathrm{z} - \frac{1-\alpha}{n-\alpha} z^n, \qquad (n \geq 2) \qquad (5.2.1.6)$$

5.2.2 Distortion properties

In this subsection, the modulus of $f(z)$ and its derivative for the class

$T^*(p,\alpha)$ according to Owa [40] are obtained.

Theorem 5.2.2.1

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Then

$$|z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1} \le |f(z)| \le |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}, \tag{5.2.2.1}$$

and

$$p|z|^{p-1} - \frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p \le |f'(z)| \le p|z|^{p-1} + \frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p. \tag{5.2.2.2}$$

for $z \in \mathcal{U}$, the results are sharp.

Proof

By virtue of Theorem 5.2.1.1, we can observe that

$$\begin{aligned}(p+1-\alpha)\sum_{n=1}^{\infty} a_{p+n} &\le \sum_{n=1}^{\infty}(p+n-\alpha)a_{p+n} \\ &\le p-\alpha\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{p-\alpha}{p+1-\alpha} \tag{5.2.2.3}$$

Hence the first estimate (5.2.2.1) follows from (5.2.2.3). Furthermore, from Theorem 5.2.1.1, we note that

$$\frac{p+1-\alpha}{p+1}\sum_{n=1}^{\infty}(p+n)a_{p+n} \le p-\alpha$$

which gives that

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le \frac{(p+1)(p-\alpha)}{p+1-\alpha} \tag{5.2.2.4}$$

Consequently, we can show the second estimate (5.2.2.2) with the aid of (5.2.2.4). Further the estimates of the theorem are sharp for the function $f(z)$ defined by

$$f(z) = z^p - \frac{p-\alpha}{p+1-\alpha} z^{p+1} \tag{5.2.2.5}$$

Corollary 5.2.2.2

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| \leq r_1$, where

$$r_1 = \frac{1}{p+1-\alpha} \tag{5.2.2.6}$$

The result is sharp with the extremal function $f(z)$ given by (5.2.2.5).

5.2.3 Convolution properties

The following Hadamard product (or convolution) properties for the class $T^*(p,\alpha)$ established by Sălăgean et. al. [56] are studied.

Theorem 5.2.3.1

Let the functions $f_i(z), (i = 1,2)$ defined by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,i}\, z^{p+n}\,, \qquad \left(a_{p+n,i} \geq 0;\ p \in \mathbb{N}\right) \tag{5.2.3.1}$$

be in the class $T^*(p,\alpha)$. Then $(f_1 * f_2)(z) \in T^*(p, \zeta(p,\alpha))$, where

$$\zeta(p,\alpha) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2} \tag{5.2.3.2}$$

The result is sharp.

Proof

We need to find the largest $\zeta = \zeta(p,\alpha)$ such that

$$\sum_{n=1}^{\infty} \frac{p+n-\zeta}{p-\zeta}\, a_{p+n,1}\, a_{p+n,2} \leq 1$$

since

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,1} \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,2} \leq 1$$

we have

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\; a_{p+n,2}} \leq 1$$

Thus it is sufficient to show that

$$\frac{p+n-\zeta}{p-\zeta}\, a_{p+n,1}\, a_{p+n,2} \leq \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\; a_{p+n,2}}$$

that is

$$\sqrt{a_{p+n,1}\; a_{p+n,2}} \leq \frac{(p-\zeta)(p+n-\alpha)}{(p-\alpha)(p+n-\zeta)}$$

Notice that

$$\sqrt{a_{p+n,1}\; a_{p+n,2}} \leq \frac{p-\alpha}{p+n-\alpha}$$

Consequently, we need only to prove that

$$\frac{p-\alpha}{p+n-\alpha} \leq \frac{(p-\zeta)(p+n-\alpha)}{(p-\alpha)(p+n-\zeta)}$$

or, equivalently, that

$$\zeta \leq p - \frac{n(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2}$$

since

$$\Psi(n) = p - \frac{n(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2} \qquad (5.2.3.3)$$

is an increasing function of n ($n \in \mathbb{N}$), letting $n = 1$ in (5.2.3.3), we obtain

$$\zeta \leq \Psi(1) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2}$$

Finally, The result is sharp for the functions

$$f_i(z) = z^p - \frac{p-\alpha}{p+1-\alpha} z^{p+1}, \qquad (i = 1,2) \qquad (5.2.3.4)$$

Theorem 5.2.3.2

Let the function $f_1(z)$ defined by (5.2.3.1) be in the class $T^*(p,\alpha)$ and let the function $f_2(z)$ defined by (5.2.3.1) be in the class $T^*(p,\vartheta)$. Then $f_1(z) * f_2(z) \in T^*\big(p, \kappa(p,\alpha,\vartheta)\big)$, where

$$\kappa = \kappa(p,\alpha,\vartheta) = p - \frac{(p-\alpha)(p-\vartheta)}{(p+1-\alpha)(p+1-\vartheta) - (p-\alpha)(p-\vartheta)} \qquad (5.2.3.5)$$

The result is sharp.

Proof

Proceeding as in the proof of Theorem 5.2.3.1, we get

$$\kappa \le \Phi(n) = p - \frac{n(p-\alpha)(p-\vartheta)}{(p+n-\alpha)(p+n-\vartheta) - (p-\alpha)(p-\vartheta)} \qquad (5.2.3.6)$$

Since the function $\Phi(n)$ is an increasing function of n ($n \in \mathbb{N}$), letting $n = 1$ in (5.2.3.6), we obtain

$$\kappa \le \Phi(1) = p - \frac{(p-\alpha)(p-\vartheta)}{(p+1-\alpha)(p+1-\vartheta) - (p-\alpha)(p-\vartheta)}$$

The result is sharp for the functions

$$f_1(z) = z^p - \frac{p-\alpha}{p+1-\alpha} z^{p+1} \qquad (5.2.3.7)$$

and

$$f_2(z) = z^p - \frac{p-\vartheta}{p+1-\vartheta} z^{p+1} \qquad (5.2.3.8)$$

5.2.4 Closure properties

The following closure theorem proven by Owa [40] is given.

Theorem 5.2.4.1

Let the functions $f_i(z), (i = 1,2, \ldots\ldots, m)$ defined by (5.2.3.1) be in the class $T^*(p,\alpha)$. Then the function

$$h(z) = \sum_{i=1}^{m} c_i \, f_i(z), \qquad\qquad (c_i \geq 0) \tag{5.2.4.1}$$

is also in the class $T^*(p, \alpha)$, where

$$\sum_{i=1}^{m} c_i = 1 \tag{5.2.4.2}$$

Proof

According to the definition of $h(z)$, we can write that

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} c_i a_{p+n,i} \right) z^{p+n}$$

By means of Theorem 5.2.1.1, we have

$$\sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n,i} \leq p - \alpha$$

for every $i = 1,2,3, \ldots\ldots, m$. Hence we can observe that

$$\begin{aligned} \sum_{n=1}^{\infty} (p+n-\alpha) \left(\sum_{i=1}^{m} c_i a_{p+n,i} \right) &= \sum_{i=1}^{m} c_i \left(\sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n,i} \right) \\ &\leq \left(\sum_{i=1}^{m} c_i \right) (p-\alpha) \\ &= p - \alpha \end{aligned}$$

which implies that $h(z) \in T^*(p, \alpha)$.

Further, the following result proven by Sălăgean et. al. [56] is given as well.

Theorem 5.2.4.2

Let the functions $f_i(z), (i = 1, 2)$ defined by (5.2.3.1) be in the class $T^*(p, \alpha)$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty} (a_{p+n,1}^2 + a_{p+n,2}^2) \, z^{p+n} \tag{5.2.4.3}$$

belongs to the class $T^*(p, \rho(p, \alpha))$, where

$$\rho(p,\alpha) = p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2} \tag{5.2.4.4}$$

The result is sharp.

Proof

By virtue of Theorem 5.2.1.1, we obtain

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,1}^2 \leq \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \; a_{p+n,1}\right\}^2 \leq 1 \tag{5.2.4.5}$$

and

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,2}^2 \leq \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \; a_{p+n,2}\right\}^2 \leq 1 \tag{5.2.4.6}$$

It follows from (5.2.4.5) and (5.2.4.6) that

$$\sum_{n=1}^{\infty} \frac{1}{2}\left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \leq 1$$

Therefore, we need to find the largest ρ such that

$$\frac{p+n-\rho}{p-\rho} \leq \frac{1}{2}\left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2$$

that is,

$$\rho \leq p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2}$$

since

$$D(n) = p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2} \tag{5.2.4.7}$$

is an increasing function of n ($n \in \mathbb{N}$), letting $n = 1$ in (5.2.4.7) we have

$$\rho \leq p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2}$$

The result is sharp for the functions $f_i(z), (i = 1,2)$ given by (5.2.3.4).

5.2.5 Extreme points

Owa [40] proved the following property which is concerned with the extreme points of the class $T^*(p,\alpha)$.

Theorem 5.2.5.1

Let

$$f_0(z) = z^p \tag{5.2.5.1}$$

and

$$f_n(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n} \tag{5.2.5.2}$$

for $0 \le \alpha < p$ and $p, n \in \mathbb{N}$, then $f(z) \in T^*(p,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_n \, f_n(z) \tag{5.2.5.3}$$

where $n \in \mathbb{N} \cup \{0\}$ and

$$\varepsilon_n \ge 0, \qquad \sum_{n=0}^{\infty} \varepsilon_n = 1 \tag{5.2.5.4}$$

Proof

Assume that

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_n \, f_n(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p-\alpha}{p+n-\alpha} \, \varepsilon_n \, z^{p+n}$$

then, we get that

$$\sum_{n=1}^{\infty} (p+n-\alpha)\left(\frac{p-\alpha}{p+n-\alpha}\right)\varepsilon_n \le p-\alpha$$

This show that $f(z) \in T^*(p,\alpha)$. Conversely, assume that $f(z) \in T^*(p,\alpha)$, by using Theorem 5.2.1.1, we can show that

$$a_{p+n} \leq \frac{p-\alpha}{p+n-\alpha}$$

for $n \in \mathbb{N}$, setting

$$\varepsilon_n = \frac{p+n-\alpha}{p-\alpha}\ a_{p+n}$$

and

$$\varepsilon_0 = 1 - \sum_{n=1}^{\infty} \varepsilon_n$$

we have the representation (5.2.5.3). This completes the proof of the Theorem 5.2.5.1.

5.2.6 Radius of convexity

In this subsection, the radius of convexity of functions in the class $T^*(p,\alpha)$ determined by Owa [40], is given as follows.

Theorem 5.2.6.1

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Then $f(z)$ is p-valently convex of order σ $(0 \leq \sigma < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_{n\in\mathbb{N}} \left\{\frac{p(p-\sigma)(p+n-\alpha)}{(p+n)(p-\alpha)(p+n-\sigma)}\right\}^{1/n} \tag{5.2.6.1}$$

with equality for a function $f(z)$ of the form (5.2.1.2).

Proof

It suffices to show that

$$\left|1 + \frac{z\,f''(z)}{f'(z)} - p\right| \leq p - \sigma, \qquad (|z| < r_2)$$

But

$$\left|1 + \frac{z\,f''(z)}{f'(z)} - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)a_{p+n}\ z^n}{p - \sum_{n=1}^{\infty}(p+n)\ a_{p+n}\ z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}\ |z|^n}{p - \sum_{n=1}^{\infty}(p+n)a_{p+n}\ |z|^n}$$

$$\leq p - \sigma$$

is true if

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n-\sigma)}{p(p-\sigma)} a_{p+n} |z|^n \leq 1$$

By Theorem 5.2.1.1, we need only to find values of $|z|$ for which

$$\frac{(p+n)(p+n-\sigma)}{p(p-\sigma)} |z|^n \leq \frac{p+n-\alpha}{p-\alpha} \tag{5.2.6.2}$$

Solving (5.2.6.2) for $|z|$, we get the desired result (5.2.6.1).

5.2.7 Class-preserving integral operators

The following results are dedicated for the class-preserving properties of the integral operator $J_{c,p}\big(f(z)\big)$ for $f(z) \in T(p)$ due to Sălăgean et. al. [56].

Theorem 5.2.7.1

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Also let $c > -p$. Then the function $F(z)$ defined by

$$F(z) = J_{c,p}\big(f(z)\big) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{5.2.7.1}$$

is also in to the class $T^*(p,\alpha)$.

Proof

From the representation of $F(z)$ it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} \mathrm{A}_{p+n} z^{p+n},$$

where

$$\mathrm{A}_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n}$$

Therefore

$$\sum_{n=1}^{\infty}(p+n-\alpha)\left(\frac{c+p}{c+p+n}\right)a_{p+n} \le \sum_{n=1}^{\infty}(p+n-\alpha)\,a_{p+n}$$
$$\le p-\alpha$$

since $f(z) \in T^*(p,\alpha)$. Hence, by Theorem 5.2.1.1, $F(z) \in T^*(p,\alpha)$.

Remark 5.2.7.1

Letting $c = 1-p$ in Theorem 5.2.7.1, the following result is obtained.

Corollary 5.2.7.2

Let the function $f(z)$ defined by (2.2.2) be in the class $T^*(p,\alpha)$. Then

$$G(z) = z^{p-1}\int_0^z \frac{f(t)}{t^p}\,dt \qquad (5.2.7.2)$$

is also in the class $T^*(p,\alpha)$.

Theorem 5.2.7.3

Let $c > -p$ and the function $F(z)$ be in the class $T^*(p,\alpha)$. Then the function $f(z)$ given by (5.2.7.1) is p-valent in the disk $|z| < r_3$ where

$$r_3 = \inf_{n\in\mathbb{N}}\left\{\frac{p(p+n-\alpha)(c+p)}{(p+n)(c+p+n)(p-\alpha)}\right\}^{1/n} \qquad (5.2.7.3)$$

The result is sharp.

Proof

Let

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\,z^{p+n}\ , \qquad \left(b_{p+n} \ge 0; p\in\mathbb{N}\right) \qquad (5.2.7.4)$$

It follows from (5.2.7.1), that

$$f(z) = \frac{z^{1-c}}{c+p}\left(z^c F(z)\right)'$$
$$= z^p - \sum_{n=1}^{\infty}\left(\frac{c+p+n}{c+p}\right)b_{p+n}\,z^{p+n}$$

in order to prove the result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p, \qquad (|z| < r_3) \tag{5.2.7.5}$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)b_{p+n}z^n\right|$$

$$\leq \sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)b_{p+n}|z|^n$$

Thus (5.2.7.5) is true if

$$\sum_{n=1}^{\infty}\frac{(p+n)(c+p+n)}{p(c+p)}b_{p+n}|z|^n \leq 1 \tag{5.2.7.6}$$

By Theorem 5.2.1.1, confirm that

$$\sum_{n=1}^{\infty}\frac{p+n-\alpha}{p-\alpha}b_{p+n} \leq 1$$

Thus (5.2.7.6) will be satisfied if

$$\frac{(p+n)(c+p+n)}{p(c+p)}|z|^n \leq \frac{p+n-\alpha}{p-\alpha}, \qquad (n \in \mathbb{N})$$

or, if

$$|z| \leq \left\{\frac{p(c+p)(p+n-\alpha)}{(p+n)(c+p+n)(p-\alpha)}\right\}^{1/n}, \qquad (n \in \mathbb{N}) \tag{5.2.7.7}$$

which leads us precisely to the main assertion of Theorem 5.2.7.3.

5.3 On a generalized class of p-valent functions

In the current section, a new class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ of analytic and p-valent starlike functions involving the generalized differential operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ given by (2.8.3.8) for $f(z) \in T(p)$ is introduced and their properties are investigated according to Amsheri and Abouthfeerah [6].

Definition 5.3.1

A function $f(\mathrm{z}) \in T(p)$ is said to be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if it satisfies

$$\frac{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \qquad (5.3.1)$$

for $p \in \mathbb{N}, m \in \mathbb{N}_0,\ \lambda \geq 0,\ \mu < p+1, \eta > \max(\lambda,\mu) - p - 1, \delta \geq 0, 0 < \beta \leq 1$ and $0 \leq \alpha < 1$. The condition (5.3.1) is equivalent to

$$\left|\frac{\dfrac{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)} - 1}{\dfrac{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)} + 1 - 2\alpha}\right| < 1, \qquad (z \in \mathcal{U}) \quad (5.3.2)$$

where $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ is given by (2.8.3.8).

Notice that, for $\lambda = \mu = \delta = 0$ and $\beta = 1$, the a-fore-mentioned class reduces to the class $T^*(p,\alpha)$ which was introduced earlier by Owa [40] and studied by Sălăgean et. al. [56]. Further, for $\lambda = \mu = \delta = 0$ and $p = \beta = 1$, it reduces to the class $T^*(\alpha)$ which was studied by Silverman [60].

5.3.1 Coefficient bounds

In this subsection, the sufficient and necessary conditions for the functions $f(z) \in T(p)$ to be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are proven.

Theorem 5.3.1.1

Let the function $f(z)$ be defined by (2.2.2). Then $f(z)$ belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p) a_{p+n} \leq p(1-\alpha) \qquad (5.3.1.1)$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6).

Proof

Since $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then

$$\left|\frac{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' - \left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)}{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' + (1-2\alpha)\left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)}\right| < 1 \tag{5.3.1.2}$$

It follows from (5.3.1.2) that

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right)-\beta\right]a_{p+n}z^n}{2(1-\alpha) - \sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right)+\beta(1-2\alpha)\right]a_{p+n}z^n}\right\} < 1$$

Choosing values of z on the real axis so that $\frac{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right)-\beta\right]a_{p+n}$$

$$\leq 2(1-\alpha) - \sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right)+\beta(1-2\alpha)\right]a_{p+n}$$

which gives the desired assertion (5.3.1.1). Conversely, let the inequality (5.3.1.1) holds true and let $|z| = 1$.Then we have

$$\left|\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' - \left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)\right|$$

$$-\left|\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' + (1-2\alpha)\left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)\right|$$

$$= \left| -\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^{m} \gamma_n(\lambda,\mu,\eta,p) \left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} z^{p+n} \right|$$

$$- \left| \begin{array}{c} 2(1-\alpha)z^p \\ -\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^{m} \gamma_n(\lambda,\mu,\eta,p) \left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} z^{p+n} \end{array} \right|$$

$$\leq 2\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^{m} \left[\left(\frac{p+n}{p}\right) - \alpha\beta\right] \gamma_n(\lambda,\mu,\eta,p) a_{p+n} - 2(1-\alpha) \leq 0$$

Hence $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. This completes the proof.

Corollary 5.3.1.2

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then

$$a_{p+n} \leq \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^{m} [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)} \tag{5.3.1.3}$$

where $p, n \in \mathbb{N}$ and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). The result (5.3.1.3) is sharp for a function of the form

$$f(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^{m} [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)} z^{p+n} \tag{5.3.1.4}$$

Remark 5.3.1.1

Letting $p = 1, \lambda = \mu = \delta = 0$ and $\beta = 1$ in Theorem 5.3.1.1 and Corollary 5.3.1.2 respectively, Corollary 5.2.1.3 and Corollary 5.2.1.4 due to Silverman [60] are obtained.

5.3.2 Distortion properties

The modulus of $f(z)$ and its derivative for the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are obtained as follows.

Theorem 5.3.2.1

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ such that $m \in \mathbb{N}_0$, $\lambda \geq 0$, $\mu < p+1$, $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$, $\delta \geq 0$, $0 < \beta \leq 1$, $0 \leq \alpha < 1$ and $p \in \mathbb{N}$, then

$$|f(z)| \geq |z|^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \tag{5.3.2.1}$$

$$|f(z)| \leq |z|^p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \tag{5.3.2.2}$$

$$|f'(z)| \geq p|z|^{p-1} - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^{p}, \tag{5.3.2.3}$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^{p}. \tag{5.3.2.4}$$

for $z \in \mathcal{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof

We observe that the function $\gamma_n(\lambda,\mu,\eta,p)$ defined by (2.8.3.6) satisfies the inequality $\gamma_n(\lambda,\mu,\eta,p) \leq \gamma_{n+1}(\lambda,\mu,\eta,p)$, $\forall n \in \mathbb{N}$, provided that $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$. Thereby, showing that $\gamma_n(\lambda,\mu,\eta,p)$ is non-decreasing. Thus under the hypothesis of the theorem, we have

$$0 < \frac{(p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)} = \gamma_1(\lambda,\mu,\eta,p) \leq \gamma_n(\lambda,\mu,\eta,p)$$

for $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, in view of Theorem 5.3.1.1, we have

$$\frac{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)}\sum_{n=1}^{\infty} a_{p+n}$$
$$\leq \sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)a_{p+n}$$
$$\leq p(1-\alpha)$$

which gives

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}$$

Consequently, we obtain

$$|f(z)| \geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$
$$\geq |z|^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}\, |z|^{p+1}$$

and

$$|f(z)| \leq \ |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$
$$\leq |z|^p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}\, |z|^{p+1}$$

which prove the assertions (5.3.2.1) and (5.3.2.2) of Theorem 5.3.2.1. Furthermore, from Theorem 5.3.1.1, we note that

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} \tag{5.3.2.5}$$

Thus, we have

$$|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty}(p+n)a_{p+n} \tag{5.3.2.6}$$

and

$$|f'(z)| \leq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty}(p+n)a_{p+n} \tag{5.3.2.7}$$

On using (5.3.2.5), (5.3.2.6) and (5.3.2.7), we arrive at the desired results (5.3.2.3) and (5.3.2.4).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} z^{p+1} \tag{5.3.2.8}$$

Corollary 5.3.2.2

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is included in a disk with center at the origin and radius r_4 given by

$$r_4 = 1 + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} \tag{5.3.2.9}$$

and $f'(z)$ is included in a disk with center at the origin and radius r_5 given by

$$r_5 = p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} \tag{5.3.2.10}$$

5.3.3 Convolution properties

In this subsection, the Hadamard product properties of any two functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are obtained.

Theorem 5.3.3.1

Let the functions $f_i(z), (i = 1,2)$ defined by (5.2.3.1) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ such that $m \in \mathbb{N}_0$, $\lambda \geq 0$, $\mu < p+1$, $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$, $\delta \geq 0$, $0 < \beta \leq 1$, $0 \leq \alpha < 1$ and $p \in \mathbb{N}$. Then $(f_1 * f_2)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\sigma)$, where

$$\sigma \leq \inf_{n\in\mathbb{N}}\left\{\frac{\nu(n)-p(p+n)(1-\alpha)^2}{\nu(n)-p^2\beta(1-\alpha)^2}\right\} \tag{5.3.3.1}$$

and

$$\nu(n) = \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]^2\gamma_n(\lambda,\mu,\eta,p) \tag{5.3.3.2}$$

Proof

It suffices to prove that

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)}\, a_{p+n,1}\, a_{p+n,2} \leq 1$$

since

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\, a_{p+n,1} \leq 1$$

and

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\, a_{p+n,2} \leq 1$$

we have

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\sqrt{a_{p+n,1}\, a_{p+n,2}} \leq 1 \tag{5.3.3.3}$$

Thus, we need to find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)} \, a_{p+n,1} \;\, a_{p+n,2}$$

$$\leq \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \sqrt{a_{p+n,1} \;\, a_{p+n,2}}$$

or, equivalently that

$$\sqrt{a_{p+n\,,1} \;\, a_{p+n\,,2}} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}$$

In view of (5.3.3.3), it is sufficient to find the largest σ such that

$$\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}$$

(5.3.3.4)

The inequality (5.3.3.4) yields

$$\sigma \leq \left\{\frac{\nu(n)-p(p+n)(1-\alpha)^2}{\nu(n)-p^2\beta(1-\alpha)^2}\right\}$$

where

$$\nu(n) = \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]^2 \gamma_n(\lambda,\mu,\eta,p)$$

which completes the proof of Theorem 5.3.3.1.

Corollary 5.3.3.2

Let the functions $f_i(z), (i=1,2)$ defined by (5.2.3.1) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1} \, a_{p+n,2}} \; z^{p+n}, \qquad (p \in \mathbb{N}) \qquad (5.3.3.5)$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof

since

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \, a_{p+n,1} \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \, a_{p+n,2} \leq 1$$

we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \sqrt{a_{p+n,1} \;\; a_{p+n,2}} \leq 1$$

By Theorem 5.3.1.1, we get $h(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Theorem 5.3.3.3

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Also, let

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} \;\; z^{p+n} \,, \qquad \left(\left|b_{p+n}\right| \leq 1; p \in \mathbb{N}\right) \qquad (5.3.3.6)$$

Then $(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof

Since

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \, \gamma_n(\lambda,\mu,\eta,p)\left|a_{p+n}b_{p+n}\right|$$

$$= \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \, \gamma_n(\lambda,\mu,\eta,p)a_{p+n}\left|b_{p+n}\right|$$

$$\leq \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \, \gamma_n(\lambda,\mu,\eta,p)a_{p+n}$$

$$\leq p(1-\alpha)$$

By Theorem 5.3.1.1, it follows that $(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

5.3.4 Closure properties

The following linear combinations of functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are proven.

Theorem 5.3.4.1

Let the functions $f_i(z),(i=1,2,\ldots\ldots m)$ defined by (5.2.3.1) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function

$$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}a_{p+n,i}\right)z^{p+n} \tag{5.3.4.1}$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof

Since $f_i(z)\in\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, $(i=1,2,\ldots\ldots m)$. By Theorem 5.3.1.1, we have

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\ a_{p+n,i}\le 1$$

so

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\left(\frac{1}{m}\sum_{i=1}^{m}a_{p+n,i}\right)=$$

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\ a_{p+n,i}\le 1$$

which shows that $h(z)\in\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Theorem 5.3.4.2

Let the functions $f_i(z),(i=1,2)$ defined by (5.2.3.1) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ such that $m\in\mathbb{N}_0$ $\lambda\ge 0$, $\mu<p+1$, $\eta\ge\lambda\left(1-\frac{p+2}{\mu}\right)$, $\delta\ge 0$, $0<\beta\le 1$, $0\le\alpha<1$ and $p\in\mathbb{N}$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty} (a_{p+n,1}^2 + a_{p+n,2}^2)\, z^{p+n} \tag{5.3.4.2}$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\sigma)$, where

$$\sigma \leq \inf_{n\in\mathbb{N}} \left\{ \frac{\nu(n) - 2p(p+n)(1-\alpha)^2}{\nu(n) - 2p^2\beta(1-\alpha)^2} \right\}, \tag{5.3.4.3}$$

and $\nu(n)$ is given by (5.3.3.2).

Proof

By virtue of Theorem 5.3.1.1, we obtain

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2 a_{p+n,1}^2 \leq$$

$$\left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \; a_{p+n,1} \right\}^2 \leq 1 \tag{5.3.4.4}$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2 a_{p+n,2}^2 \leq$$

$$\left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \; a_{p+n,2} \right\}^2 \leq 1 \tag{5.3.4.5}$$

It follows from (5.3.4.4) and (5.3.4.5) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2 \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \leq 1$$

Therefore, we need to find the largest σ such that

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^{m}[n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)}\left(a_{p+n,1}^2+a_{p+n,2}^2\right)\leq 1$$

Thus, it is sufficient to show that

$$\frac{\left(\frac{p+\delta n}{p}\right)^{m}[n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)}$$

$$\leq \frac{1}{2}\left\{\frac{\left(\frac{p+\delta n}{p}\right)^{m}[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\right\}^2 \qquad (5.3.4.6)$$

The inequality (5.3.4.6) yields

$$\sigma \leq \left\{\frac{\nu(n)-2p(p+n)(1-\alpha)^2}{\nu(n)-2p^2\beta(1-\alpha)^2}\right\}$$

where $\nu(n)$ is given by (5.3.3.2). This completes the proof of the Theorem 5.3.4.2.

Theorem 5.3.4.3

The class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ is convex.

Proof

Suppose that the functions $f_i(z),(i=1,2)$ defined by (5.2.3.1) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then it is sufficient to show that the function

$$h(z)=\xi f_1(z)+(1-\xi)f_2(z), \qquad (0\leq\xi\leq 1)$$

or, equivalently

$$h(z)=z^p-\sum_{n=1}^{\infty}\left\{\xi a_{p+n,1}+(1-\xi)a_{p+n,2}\right\}z^{p+n}, \qquad (0\leq\xi\leq 1)$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Now, from our hypothesis and Theorem 5.3.1.1, it follows readily that

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^{m}[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)\ \left(\xi a_{p+n,1}+(1-\xi)a_{p+n,2}\right)$$
$$\leq p(1-\alpha)$$

which evidently proves Theorem 5.3.4.3.

5.3.5 Extreme points

The extreme points for the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ can be determined as follows.

Theorem 5.3.5.1

Let

$$f_p(z)=z^p \qquad (5.3.5.1)$$

and

$$f_{p+n}(z)=z^p-\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^{m}[n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)}z^{p+n} \qquad (5.3.5.2)$$

for $p,n\in\mathbb{N}$. Then $f(z)\in\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\varepsilon_{p+n}\ f_{p+n}(z) \qquad (5.3.5.3)$$

where

$$\varepsilon_{p+n}\geq 0,\qquad \sum_{n=0}^{\infty}\varepsilon_{p+n}=1 \qquad (5.3.5.4)$$

Proof

Let

$$f(z)=\sum_{n=0}^{\infty}\varepsilon_{p+n}\ f_{p+n}(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}\, \varepsilon_{p+n}\, z^{p+n}$$

Then, in view of (5.3.5.4), it follows that

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right.$$

$$\left. \times \left(\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)} \varepsilon_{p+n} \right) \right\}$$

$$= \sum_{n=1}^{\infty} \varepsilon_{p+n} = 1 - \varepsilon_p \leq 1$$

Therefore, $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Conversely, suppose that $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then

$$a_{p+n} \leq \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}, \qquad (n \in \mathbb{N})$$

Setting

$$\varepsilon_{p+n} = \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\, a_{p+n}\,, \qquad (n \in \mathbb{N})$$

and

$$\varepsilon_p = 1 - \sum_{n=1}^{\infty} \varepsilon_{p+n}$$

we can see that $f(z)$ can be expressed in the form (5.3.5.3). This completes the proof of the Theorem 5.3.5.1.

Corollary 5.3.5.2

The extreme points of the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are the functions $f_p(z)$ and $f_{p+n}(z)$ given by (5.3.5.1) and (5.3.5.2) respectively.

5.3.6 Radii of close-to-convexity, starlikeness and convexity

Firstly, the largest disk $|z| < r_6$, $0 < r_6 \leq 1$ for functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ to be close-to-convex in $|z| < r_6$ is determined as follows.

Theorem 5.3.6.1

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently close-to-convex of order σ, $0 \leq \sigma < p$ in the disk $|z| < r_6$, where

$$r_6 = \inf_{n\in\mathbb{N}} \left\{ \frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)(p+n)} \right\}^{1/n} \qquad (5.3.6.1)$$

and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). The result is sharp with the extremal function $f(z)$ given by (5.3.1.4).

Proof

It suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \sigma, \qquad (|z| < r_6) \qquad (5.3.6.2)$$

Indeed, we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=1}^{\infty} (p+n) a_{p+n} |z|^n$$

Hence (5.3.6.2) is true if

$$\sum_{n=1}^{\infty} \frac{p+n}{p-\sigma} a_{p+n} |z|^n \leq 1 \qquad (5.3.6.3)$$

By Theorem 5.3.1.1, and (5.3.6.3) is true if

$$\frac{p+n}{p-\sigma}|z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \qquad (5.3.6.4)$$

Solving (5.3.6.4) for $|z|$ $(n \in \mathbb{N})$, we get the desired result (5.3.6.1).

Next, the largest disk $|z| < r_7,\ 0 < r_7 \leq 1$ for functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ to be starlike in $|z| < r_7$ is determined as follows.

Theorem 5.3.6.2

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently starlike of order $\sigma, 0 \leq \sigma < p$ in the disk $|z| < r_7$, where

$$r_7 = \inf_{n\in\mathbb{N}} \left\{ \frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)(p+n-\sigma)} \right\}^{1/n} \tag{5.3.6.5}$$

and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). The result is sharp with the extremal function $f(z)$ given by (5.3.1.4).

Proof

It suffices to show that

$$\left|\frac{z\,f'(z)}{f(z)} - p\right| \leq p - \sigma, \qquad (|z| < r_7) \tag{5.3.6.6}$$

Indeed, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n a_{p+n}\, z^n}{1 - \sum_{n=1}^{\infty} a_{p+n}\, z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n a_{p+n} |z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n}\, |z|^n}$$

Hence (5.3.6.6) is true if

$$\sum_{n=1}^{\infty} n a_{p+n}\, |z|^n \leq (p-\sigma) - \sum_{n=1}^{\infty} (p-\sigma) a_{p+n}\, |z|^n$$

that is, if

$$\sum_{n=1}^{\infty} \frac{p+n-\sigma}{p-\sigma} a_{p+n} \, |z|^n \ \leq 1 \tag{5.3.6.7}$$

By Theorem 5.3.1.1, and (5.3.6.7) is true if

$$\frac{p+n-\sigma}{p-\sigma} \ |z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m \ [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \tag{5.3.6.8}$$

Solving (5.3.6.8) for $|z|$ $(n \in \mathbb{N})$, we get the desired result (5.3.6.5).

Finally, the largest disk $|z| < r_8$, $0 < r_8 \leq 1$ for functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ to be convex in $|z| < r_8$ is determined as follows.

Theorem 5.3.6.3

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently convex of order σ, $0 \leq \sigma < p$ in the disk $|z| < r_8$, where

$$r_8 = \inf_{n\in\mathbb{N}} \left\{ \frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m \ [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(1-\alpha)(p+n-\sigma)} \right\}^{1/n}$$

(5.3.6.9)

and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6). The result is sharp with the extremal function $f(z)$ given by (5.3.1.4).

Proof

It suffices to show that

$$\left|1+\frac{z\,f''(z)}{f'(z)}-p\right| \leq p-\sigma, \qquad (|z| < r_8) \tag{5.3.6.10}$$

Indeed, we have

$$\left|1+\frac{z\,f''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)a_{p+n} \ z^n}{p-\sum_{n=1}^{\infty}(p+n)\ a_{p+n} \ z^n}\right|$$

$$\leq \ \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}\,|z|^n}{p-\sum_{n=1}^{\infty}(p+n)a_{p+n}\,|z|^n}$$

Hence (5.3.6.10) is true if

$$\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}\, |z|^n \leq p(p-\sigma) - \sum_{n=1}^{\infty} (p-\sigma)(p+n) a_{p+n}\, |z|^n$$

or

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n-\sigma)}{p(p-\sigma)} a_{p+n}\, |z|^n \leq 1 \qquad (5.3.6.11)$$

By Theorem 5.3.1.1, and (5.3.6.11) is true if

$$\frac{(p+n)(p+n-\sigma)}{(p-\sigma)}\, |z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(1-\alpha)}$$

(5.3.6.12)

Solving (5.3.6.12) for $|z|$ $(n \in \mathbb{N})$, we get the desired result (5.3.6.9).

5.3.7 Class-preserving integral operators

In the present subsection, further properties of the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ of functions under the generalized Bernardi-Libera-Livingston integral operator defined in (5.2.7.1) are investigated. The closure property can be proven as follows.

Theorem 5.3.7.1

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Also, let $c > -p$. Then the function $F(z)$ defined by (5.2.7.1) is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof

From (5.2.7.1) and (2.2.2), we have

$$F(z) = z^p - \sum_{n=1}^{\infty} A_{p+n} z^{p+n}$$

where

$$A_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n}\,, \qquad (n \in \mathbb{N})$$

Since $c > -p$, we have

$$0 \leq A_{p+n} < a_{p+n}, \qquad (n \in \mathbb{N})$$

which, in view of Theorem 5.3.1.1, immediately yields Theorem 5.3.7.1.

Remark 5.3.7.1

Letting $c = 1 - p$ in Theorem 5.3.7.1, the following result is obtained.

Corollary 5.3.7.2

Let the function $f(z)$ defined by (2.2.2) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then

$$G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p}\, dt \qquad (5.3.7.1)$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Next, the largest disk $|z| < r_9$, $0 < r_9 \leq 1$ for functions in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ to be p-valent in $|z| < r_9$ is determined as follows.

Theorem 5.3.7.3

Let $c > -p$ and the function $F(z)$ be in class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function $f(z)$ given by (5.2.7.1) is p-valent in the disk $|z| < r_9$, where

$$r_9 = \inf_{n\in\mathbb{N}} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m (c+p)[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(c+p+n)(1-\alpha)} \right\}^{1/n} \qquad (5.3.7.2)$$

Proof

Let $F(z)$ defined by (5.2.7.4). It follows from (5.2.7.1), that

$$f(z) = \frac{z^{1-c}}{c+p}\frac{d}{dz}\left(z^c F(z)\right)$$

$$= z^p - \sum_{n=1}^{\infty} \left(\frac{c+p+n}{c+p}\right) b_{p+n}\, z^{p+n}, \qquad (c > -p)$$

In order to prove the result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p, \qquad (|z| < r_9) \qquad (5.3.7.3)$$

Indeed we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)b_{p+n}z^n\right|$$

$$\leq \sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right)b_{p+n}|z|^n$$

which yields the desired inequality in (5.3.7.3), provided that

$$\sum_{n=1}^{\infty}\frac{(p+n)(c+p+n)}{p(c+p)}b_{p+n}|z|^n \leq 1 \tag{5.3.7.4}$$

since the function $F(z)$ defined by (5.2.7.4) is in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then by Theorem 5.3.1.1, we have

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}b_{p+n} \leq 1$$

Thus the inequality (5.3.7.4) will hold true if

$$\frac{(p+n)(c+p+n)}{p(c+p)}|z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}$$

that is, if

$$|z| \leq \left\{\frac{\left(\frac{p+\delta n}{p}\right)^m (c+p)[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(c+p+n)(1-\alpha)}\right\}^{1/n}, \qquad (n \in \mathbb{N})$$

which leads us precisely to the main assertion of Theorem 5.3.7.3.

5.3.8 Integral means inequalities

Application of Lemma 5.1 leads to the following integral means inequality theorem for functions belonging to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Theorem 5.3.8.1

Let $\tau > 0,\ \lambda \geq 0,\ \mu < p+1,\ \eta \geq \lambda\left(1-\frac{p+2}{\mu}\right),\ \delta \geq 0,\ m \in \mathbb{N}_0\,,\ 0 < \beta \leq 1,$ $0 \leq \alpha < 1$ and $p \in \mathbb{N}$. If $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then for $z = re^{i\theta}$ and $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\tau} d\theta \leq \int_0^{2\pi} \left|f_{p+1}(re^{i\theta})\right|^{\tau} d\theta \qquad (5.3.8.1)$$

where

$$f_{p+1}(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\, \gamma_1(\lambda,\mu,\eta,p)} z^{p+1} \qquad (5.3.8.2)$$

and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (2.8.3.6).

Proof

Let $f(z)$ of the form (2.2.2) and $f_{p+1}(z)$ of the form (5.3.8.2), then we must show that

$$\int_0^{2\pi} \left|1-\sum_{n=1}^{\infty} a_{p+n} z^n\right|^{\tau} d\theta$$

$$\leq \int_0^{2\pi} \left|1-\frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\, \gamma_1(\lambda,\mu,\eta,p)} \mathrm{z}\right|^{\tau} d\theta.$$

By Lemma 5.1, it suffices to show that

$$1-\sum_{n=1}^{\infty} a_{p+n} z^n \prec 1-\frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\, \gamma_1(\lambda,\mu,\eta,p)} \mathrm{z}\,.$$

Setting

$$1-\sum_{n=1}^{\infty} a_{p+n} z^n = 1-\frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\, \gamma_1(\lambda,\mu,\eta,p)} w(z)$$

(5.3.8.3)

from (5.3.8.3) and (5.3.1.1), we obtain

$$|w(z)| = \left| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta}{p}\right)^m [1 + p(1-\beta\alpha)]\, \gamma_1(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n} z^n \right|$$

$$\leq |z| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n}$$

$$\leq |z|$$

$$< 1$$

which completes the proof.

Remark 5.3.8.1

Letting $\lambda = \mu = \delta = 0$ and $p = \beta = 1$ in Theorem 5.3.8.1, the integral means inequality for the class $T^*(\alpha)$ is obtained as follows.

Corollary 5.3.8.2

Let $\tau > 0$ and $0 \leq \alpha < 1$. If $f(z) \in T^*(\alpha)$, then for $z = re^{i\theta}$and $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \leq \int_0^{2\pi} \left|f_2\left(re^{i\theta}\right)\right|^\tau d\theta \tag{5.3.8.4}$$

where

$$f_2(z) = z - \frac{1-\alpha}{2-\alpha} z^2 \tag{5.3.8.5}$$

Remark 5.3.8.2

Letting $\alpha = 0$ in Corollary 5.3.8.2, the following integral means result for the class T due to Silverman [61] is obtained.

Corollary 5.3.8.3

Let $\tau > 0$ and $f(z) \in T$. Then for $z = re^{i\theta}$and $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\tau d\theta \leq \int_0^{2\pi} |f_2(z)|^\tau d\theta \tag{5.3.8.6}$$

where

$$f_2(z) = z - \frac{1}{2} z^2 \tag{5.3.8.7}$$

Conclusion

The history of starlike functions goes back to 1915, when this class was studied by Alexander [1]. The present researcher is mainly concerned with a study on **some geometrical and analytical properties for certain classes of multivalent starlike functions**.

The researcher has achieved this goal by studying some basic concepts of analytic univalent and multivalent functions defined in the open unit disk and some of their related subclasses. Also, some linear operators are presented. Moreover, the technique of subordination was employed to introduce certain subclasses of the class $A(p)$ of p-valent functions in order to obtain the bounds of the coefficient functional $\left|a_{p+2} - \theta a_{p+1}^2\right|$. At this place, the well-known class $S_{b,p}^*(\phi)$ and it's a generalized class $S_{b,p,\lambda,\mu,\eta}^*(\phi)$ defined by the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ are studied. Further, a new extended class $S_{b,p,\lambda,\mu,\eta,\delta}^m(\phi)$ of p-valent functions associated with the linear operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ is introduced. Furthermore, sufficient conditions for starlikeness and convexity are obtained by using different techniques. At this place, many well-known conditions for p-valent functions associated with the operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ are studied. Also, some new conditions for p-valent functions involving the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ are investigated. In addition, certain subclasses of the class $T(p)$ of analytic and p-valent functions with negative coefficients are defined and studied to investigate coefficient bounds, distortion properties, convolution properties, closure properties, extreme points, radius of close-to-convexity, radius of starlikeness, radius of convexity, class-preserving integral operators and integral means inequalities. At this place, the well-known class $T^*(p, \alpha)$ is studied and a new generalized class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ associated with a certain linear operator is introduced and studied.

The mentioned above classes showed that the functions of these classes generalize the concept of starlike functions. For various values of the parameters, these classes reduced to classes of starlike functions.

Overall, the researcher reached the following results:

1. The careful research carried out earlier and in this thesis shows that the linear operators have many extensive and interesting applications in the theory of analytic and multivalent functions.
2. Some well-know results are reduced as a special case from the main results signifying the work presented in this thesis.

Future work

Through the results reached in this study, the researcher recommends the following:

1- A number of problems of this type of study may be raised for various linear operators. For example, the operators of Ruscheweyh, Komatu, Sălăgean and others.

2- The operators which were used in this study may be applied for other fields of analytic functions such as Harmonic functions, meromorphic functions and others.

3- Fractional calculus operators may be used for other fields of science such as partial differential equations, physics, engineering and others.

Some suggested areas of research include

1- A study on some classes of analytic functions associated with different linear operators.

2- Differential subordination and supeordination.

3- Analytic functions with negative coefficients.

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