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CERTAIN PROBLEMS RELATED TO CLASSES OF ANALYTIC FUNCTIONS AND THEIR APPLICATIONS

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بعض المشاكل المتعلقة بفصول الدوال التحليلية وتطبيقاتها

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﴿ اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ*خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ*اقْرَأْ
وَرَبُّكَ الْأَكْرَمُ*الَّذِي عَلَّمَ بِالْقَلَمِ*عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ ﴾.

Abstract

In this study, we used a generalized derivative operator, defined on new classes of analytic functions, to obtain results.

This study is consists of the following chapters:

In Chapter One, we have gathered the essential definitions and fundamental theories which are necessary for this study.

In Chapter Two, there are certain uses for fractional calculus in science, particularly in mathematics. Using the ideas of fractional calculus, we explore several applications of generalized derivative operator in the context of geometric function theory. In this chapter, we introduce a new class of analytic and normalized functions using a generalized derivative operator in the open disk.

In Chapter Three, we obtain the Fekete-Szego inequality using the generalized derivative operator and the Hadamard product, for a new class of analytic and normalized functions in the unit disk.

In Chapter Four, we study the bounded coefficients of Taylor–Maclaurin $|a_2|$ and$|a_3|$ in the open unit disk, and present two new subclasses of analytic, univalent and bi-univalent functions related to the generalized derivative operator.

الملخص

في هذه الدراسة، استخدمنا المؤثر التفاضلي المعمم، ثم تعريفه على فصول جديدة من الدوال التحليلية، لإيجاد النتائج.

تحتوي هذه الدراسة على الفصول التالية:

في الفصل الأول، جمعنا التعريفات والنظريات الأساسية اللازمة لهذه الدراسة.

في الفصل الثاني، هناك استخدامات معينة للتفاضل والتكامل الكسري في العلوم، وخاصة في الرياضيات. باستخدام أفكار حساب التفاضل والتكامل الكسري، نستكشف عدة تطبيقات للمؤثر التفاضلي المعمم في سياق نظرية الدوال الهندسية. في هذا الفصل، نقدم فصل جديد من دوال التحليلية ومعيارية باستخدام المؤثر التفاضلي المعمم في قرص الوحدة.

في الفصل الثالث، نحصل على متباينة فيكيت-سيجو باستخدام المؤثر التفاضلي المعمم مع الضرب هادا مارد، لفصل جديد من الدوال التحليلية والمعيارية في قرص الوحدة.

في الفصل الرابع، ندرس المعاملات المحدودة لتايلور وما كلورين $|a_2|$ و $|a_3|$ في قرص الوحدة، ونقدم فصلين جزئيين جديدين من الدوال التحليلية، الدوال أحادية القيمة والدوال ثنائية القيمة المرتبطة بمؤثر التفاضلي المعمم.

Acknowledement

Thanks and praise to the Almighty Allah who helped me produce this work.

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Deication

I dedicate the fruit of my labor to the soul of my brother (Tahir Abdullah), who was the main reason for my achievement in higher education after God Almighty**.** to those who endowed me with life, hope and growing up with a passion for familiarity and knowledge ,and who taught me to rise the ladder of life wisely,patiently, sensibly and loyalty to my father and mother**,** to whom God built the blessing of your existence in my life to whom they have helped me so many obstacles and difficulties during a research journey, my brothers and sister.

List of Symbols

$\mathbb{U}$	The open unit disk
$\mathbb{C}$	The complex plane
$\mathcal{K}(z)$	The Koebe function
$\mathcal{A}$	The class of analytic and normalized functions
S	The class of analytic, normalized and univalent functions
$w(z)$	Schwarz function
$\prec$	Subordinate to
$f * g$	Hadamard product (or convolution)
$\mathcal{C}$	The class of convex functions
S^*	The class of starlike function
$\mathcal{C}(\beta)$	The class of convex functions of order β
$S^*(\beta)$	The class of starlike functions of order β
$\mathcal{P}$	The class of functions with positive real part
$\mathcal{T}$	The class of bounded turning functions
Σ	The class of bi-univalent function
$\mathcal{C}_\Sigma(\beta)$	The class of bi-convex functions of order β
$S_\Sigma^*(\beta)$	The class of bi-starlike functions of order β
$S_\lambda^n f(z)$	Al-Oboudi derivative operator
$S^n f(z)$	Salagean derivative operator
$I^m(\lambda_1, \lambda_2, l, n) f(z)$	The generalized derivative operator
$(c)_k$	The pochhammer symbol (or the shifted factorial)

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Some Preliminary Concepts of Geometric Function Theory of a Complex Variable

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Introduction

Geometric function theory (GFT) is one of the most important branches of complex analysis. It was established around the 20th century and has remained an active field of research. GFT is concerned with the study of the geometric properties of analytical functions in complex analysis and has many applications in various fields of mathematics, including special functions and fractional calculus. In recent years, there has been remarkable progress in the theory of geometric functions and their various applications. The theory of univalent function is a beautiful subject as we can see in recent years, many new articles have been written in this area. This field which is often associated with ’geometry’ and ’analysis’ has raised the interest of many since the beginning of the 20th century to recent times. (Duren, 2001) The first study of univalent functions conducted by P. Koebe was published in 1907. Throughout the last 100 years and until today, univalent functions theories have developed greatly. In 1914, Gronwall‘s study proved the area theorem. In 1916, Bieberbach's estimated the second coefficient of a normalized univalent function, which was solved by Louis de Branges in 1984. (Goodman, 1983)

The study of fractional calculus, has gained considerable popularity and importance over the past four decades, due, mainly, to its demonstrated applications in numerous diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of Mathematical Physics and Applied Mathematics as well as their extensions and generalizations for one and more variables. (Srivastava, 2018) The theory of fractional calculus has been applied to theory of analytic functions. Fractional differential equations are emerging as a new and famous branch of applied mathematies that is being used for many mathematical models in science and engineering. In fact, fractional differential equations are viewed as an alternative model to nonlinear differential equations. (Khan et al., 2023)

In this study, by using the operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$, we define a new subclasses of analytic functions, find some applications for this function, estimate coefficients for the Fekete-Szego inequality, and find coefficient bounds for the functions in the two new subclasses of bi-univalent functions.

Chapter One

Some Preliminary Concepts of Geometric Function Theory of a Complex Variable

Introduction.

A function $f\ (z)$ that is analytic in the open unit disk $\mathbb{U}$ is said to be univalent in $\mathbb{U}$ if the conditions

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2 \quad ; z_1, z_2 \in \mathbb{U}.$$

Univalent function theory is a new area of great interest in GFT, which has branched out to include many fields, such as classes of bi-univalent functions, starlike and convex functions, and other many classes, which have geometric properties of analytic functions. (Duren, 2001)

The name univalent functions or schlicht (the German word for simple) functions is given to functions defined on the open unit disk $\mathbb{U} = \{z: |z| < 1\}$. (Goodman, 1983) If f is analytic and univalent function in D, then without loss of generality we can assume that is the unit disk $\mathbb{U}$ with Taylor expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The normalization $f(0) = f'(0) - 1 = 0$ is obtained by the transformation

$$\frac{f(z) - f(0)}{f'(0)}.$$

We denote by $\mathcal{A}$, the class of normalized and analytic functions $f$in $\mathbb{U}$, and the class of normalized functions that are analytic ,and univalent in $\mathbb{U}$,is denoted by S. (Thomas et al., 2018).

This chapter aims to systematically introduce the necessary details, definitions, and other preliminaries for some basic concepts in the theory of analytic functions in a complex plane. The definitions of analytic functions, univalent functions, starlike functions, convex functions, are introduced. Also, the growth and distortion theorems are constructed. Finally, some operators of the analytic functions in unit disk are given.

1.2 Basic definitions and some properties of the class of univalent functions.

Definition 1.2.1

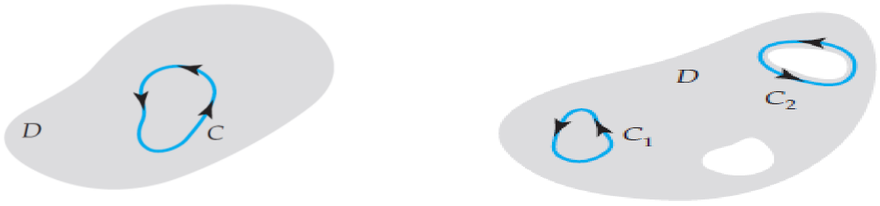
A set of points in the complex plane is connected if any two points of the set can be connected by a continuous piecewise smooth curve, all of which belong to the set. (Goodman, 1983)

Definition 1.2.2

A domain is an open subset D of the complex plane $\mathbb{C}$ that is connected. (Goodman, 1983)

A domain D is called a simply connected domain if any simple closed curve in a domain D encircles only points of D (in other words D has no holes). (Silverman, 1984)

A domain that is not simply connected is called a multiply connected domain, that is, a multiply connected domain has "holes" in it. (Silverman, 1984) (See Figure 1.1).



A domain D is Simply connected

A domain D is not simply connected (Multiply connected)

Figure 1.1:Domain D simply connected and multiply connected

Definition 1.2.3

A complex-valued function f of a complex variable z is differentiable at a point $z_0 \in \mathbb{C}$ if it has derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}.$$

at z_0 exits. (Duren, 2001)

Definition 1.2.4

The set of points inside the circle of radius r centered at the point z_0, that satisfies the inequality $|z - z_0| < r$ is called an open disk or neighborhood of z_0. (Goodman, 1983)

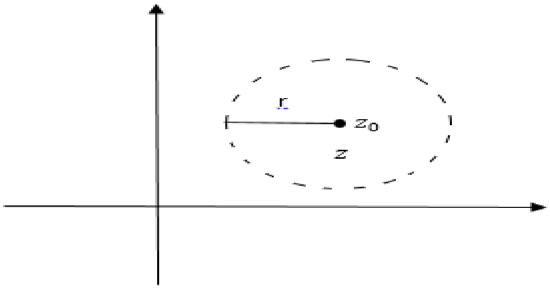


Figure 1.2: r- the neighborhood of z_0.

Definition 1.2.5 "Analytic function"

A function f of the complex variable z is said to be analytic at a point z_0 if it is differentiable at z_0 and every point in some neighborhood of z_0. A function f is analytic in a domain D if it is analytic at every point in D. (Duren, 2001)

Definition 1.2.6 " Taylor series"

The function f must have derivatives of all orders at z_0 and that has a Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,\ a_k = \frac{f^{(k)}(z_0)}{k!}, (k = 0,1,2, \dots),$$

$for\ all\ z \in \mathbb{U}_r(z_0)$, ($\mathbb{U}_\mathrm{r}(\mathrm{z}_0) = |\mathrm{z} - \mathrm{z}_0| < \mathrm{r}$).

which converges in some open disk centered at z_0. (Brown & Churchill, 2009)

Definition 1.2.7 "Maclaurin series ''

Suppose that $f(z)$ is analytic. The power series that represent a function $f(z)$ within $|\mathrm{z}| = \mathrm{r}$ (with center $\mathrm{z}_0 = 0$) that is defined by

$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\ z^k$is called the Maclaurin series. (Brown & Churchill, 2009)

Definition 1.3.8 "Conformal mapping"

Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane, for any z ∈ D for which $f'(\mathrm{z}) \neq 0$, this mapping is conformal, that is: it preserves the angle between two differentiable arcs. (Duren, 2001)

Example 1.2.1

1) The identity $f(z) = z$ is univalent in $\mathbb{U}$.(Thomas et al., 2018)

Clearly:$f(z_1) = f(z_2) \Rightarrow z_1 = z_2$

2) $f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$, which maps $\mathbb{U}$ onto the half-plane

$\left\{Re\{f(z)\} > \frac{-1}{2}\right\}$.

Suppose that

$$f(z_1) = f(z_2),$$

then,

$$\frac{z_1}{1 - z_1} = \frac{z_2}{1 - z_2}$$

$$z_1 - z_1 z_2 = z_2 - z_1 z_2$$

$$z_1 = z_2.$$

Then we give the geometry of the function f.

Let $z = x + iy$,

$$f(z) = \frac{z}{1 - z} = \frac{x + iy}{1 - x - iy} . \frac{1 - x + iy}{1 - x + iy}$$

$$Re\{f(z)\} = \frac{x - (x^2 + y^2)}{(1 - x)^2 + y^2}$$

$$> \frac{x - 1}{2 - 2x} > -\frac{1}{2}.$$

Then the function $f(z) = \frac{z}{1-z}$ is univalent on the half-plane $\left\{Re\{f(z\} > \frac{-1}{2}\right\}$.

(Thomas et al., 2018)

Example 1.2.2

The Koebe function $\mathcal{K}(z)$ is in the class S

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots + kz^k + \cdots = \sum_{k=1}^{\infty} k\, z^k, \text{ for } |z| < 1.$$

We can write the koebe function $\mathcal{K}(z)$ as the following

$$(u_3 \circ u_2 \circ u_1)(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{1}{4}\left[\frac{4z}{(1-z)^2}\right] = \frac{z}{(1-z)^2},$$

where

$$u_1(z) = \frac{1+z}{1-z}, \ \ u_2(z) = z^2, \ \ and \ \ u_3(z) = \frac{1}{4}[z-1]. \text{ (Goodman, 1983)}$$

Now if we show the functions u_1,u_2 and u_3 are analytic and univalent functions, then this composition are analytic and univalent functions.

First, we prove the function u_1 is analytic and univalent Suppose that

$$u_1(z_1) = u_1(z_2),$$

then,

$$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}$$

$$(1+z_1)\,(1-z_2) = (1-z_1)(1+z_2)$$

$$1 + z_1 - z_2 - z_1 z_2 = 1 + z_2 - z_1 - z_1 z_2$$

$$2z_1 = 2z_2 \Rightarrow \ z_1 = z_2 .$$

Therefore, $u_1(z)$ is an univalent and analytic function.

Second, we give the geometry of the function $u_1(z)$.

Let $z = x + iy$, then we have

$$u_1(z) = \frac{1+x+iy}{1-x-iy} \cdot \frac{1-x+iy}{1-x+iy}$$

$$u_1(z) = \frac{1-x^2-y^2}{(1-x)^2+y^2} + i\,\frac{2y}{(1-x)^2+y^2},$$

then

$$Re\{u_1(z)\} = \frac{1-(x^2+y^2)}{(1-x)^2+y^2} > 0\,, since \ \ x^2+y^2 < 1.$$

Thus, $u_1(z)$ maps from the unit disk into the positive real plane $Re\ \{z\} > 0$.

Next, the function $u_2(z) = z^2, Re\{z\} > 0$, is univalent and analytic.

Finally, we prove the function u_3 is an analytic and univalent function.

Suppose that

$$u_3(z_1) = u_3(z_2)$$

$$\frac{1}{4}(z_1 - 1) = \frac{1}{4}(z_2 - 1).$$

Then

$$z_1 = z_2.$$

Hence, u_3 is univalent in the entire complex plane minus the nonnegative real axis.

Then the Koebe function maps from the unit disk to the complement of the ray (-∞, $-\frac{1}{4}$].

(See Figure 1.3)

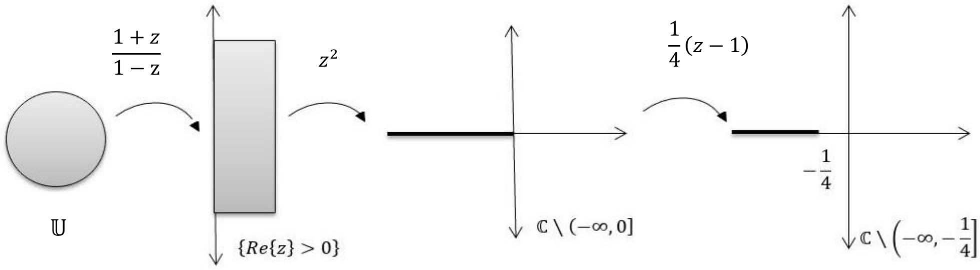


Figure 1.3: The Koebe function

Note that:

1) The koebe function $\mathcal{K}(z) = \frac{z}{(1-z)^2}$ is analytic and univalent function.

Also $\mathcal{K}(0) = 0$ and $\mathcal{K}'(0) = 1$, therefore the koebe function is in S.

2) The koebe function plays a very important role in the study of S. It is often the external function for various problems in the class S.

Definition 1.2.9 "Hadamard product"

The Hadamard product (or convolution) of two analytic functions f and g, where f is given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k \mathrm{z}^k, (z \in \mathbb{U}),$$

denoted by $f * g$ is defined by

$$(f * g)(z) = f(z) * g(z) = z + \textstyle\sum_{k=2}^{\infty} a_k b_k\, z^k. (z \in \mathbb{U}).$$ (Duren, 1983)

Lemma 1.2.1 "the Schwarz s Lemma''

An analytic function in $\mathbb{U}$ with the properties $w(0) = 0$ and $|w(z)| \leq 1$is called a Schwarz function w(z). (Duren, 2001)

Definition 1.2.10 "Subordination"

For analytic functions f and g on $\mathbb{U}$, we say that f is subordinate to g denoted $f \prec g$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ such that

$$f(z) = g\big(w(z)\big), \ \mathrm{z} \in \mathbb{U}.$$ (Goodman, 1983)

Example 1.2.2

The function z^4 is subordination to z^2 in $\mathbb{U}$ exist the schwarz function

$w(z) = z^2$in $\mathbb{U}$, satisfy $w(z^2) = w(0) = 0, |w(z)| = |z^2| = |z|^2 < 1; z \in \mathbb{U}$.

Example 1.2.3

In general, if n positive integral, then $z^n \prec z \in \mathbb{U}$ and $z^{2n} \prec z^2 \in \mathbb{U}$. (Goodman, 1983)

Theorem 1.2.1 "Bieberbach Theorem"

If $f \in S$ given by (1.1), then $|a_2| \leq 2$ with equality if and only if f is a rotation of the Koebe function. (Goodman, 1983)

Conjecture Bieberbach 1.2.2

Let the function $f \in S$, where f given by (1.1) with$f\,(0) = 0$ and$f\,'(0) = 1$. Then $|a_k| \leq k$ for all $k \geq 2$ and if there is an integer k such that$|a_k| = k$, then$f$ is a rotation of the Koebe function. In fact, if f is a rotation of the Koebe function, then $|a_k| = k$ for all k . (Goodman, 1983)

Cases found by researchers earlier are as follows:

Researcher	**Result**
Bieberbach (1916)	$\vert a_2\vert \leq 2$
Löwner (1923)	$\vert a_3\vert \leq 3$
Garabedian and Schiffer (1955)	$\vert a_4\vert \leq 4$
Pederson and Schiffer (1972)	$\vert a_5\vert \leq 5$
Ozawa(1969)	$\vert a_6\vert \leq 6$
de Branges (1984)	$\vert a_n\vert \leq n$

Table 1.1: The coefficient that were resolved

Some Important Properties of The Univalent Function.

Theorem 1.3.1

Let f be a univalent function in $\mathbb{U}$ then;

1. $g(z) = f(z) + c$ is univalent, for any $c \in \mathbb{C}$.
2. $g(z) = \lambda f(z)$ is univalent, for any $\lambda \in \mathbb{C}, \lambda \neq 0$. (Goodman,1983)

Proof

Suppose f is a univalent function in $\mathbb{U}$ and $c, \lambda \in \mathbb{C}$.

1. If $g(z) = g(w)$ then $f(z) + c = f(w) + c$ and then we get $f(z) = f(w)$.

But f is univalent then $z = w$ and so $g(z)$ is univalent.

2. If $g(z) = g(w)$ then $\lambda f(z) = \lambda f(w)$. Since $\lambda \neq 0$ we get $f(z) = f(w)$.

But f is univalent then $z = w$ and so $g(z)$ is univalent.

Theorem 1.3.2" Distortion Theorem"

For each $f \in S$,

$$\frac{1-r}{(1+r)^3} = \mathcal{K}'(-r) \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} = \mathcal{K}'(r), \qquad |z| = r < 1,$$

for each $z \in \mathbb{U}, z \neq 0$, and equality occurs if and only if a function f is a rotation of the

Koebe function. (Goodman, 1983)

Theorem 1.3.3 "Growth Theorem"

For each $f \in S$,

$$\frac{r}{(1+r)^2} = -\mathcal{K}(-r) \leq |f(z)| \leq \frac{r}{(1-r)^2} = \mathcal{K}(r), \quad |z| = r < 1,$$

for each $z \in \mathbb{U}, z \neq 0$, and equality occurs if and only if a function f is a suitable rotation of the Koebe function. (Goodman, 1983)

Special Classes of Analytic Univalent Functions in the Unit Disk.

In this section, we define special classes of analytic univalent functions in the open unit disk, the classes starlike functions and convex functions. In addition, we defined other subclasses of univalent functions.

1.4.1 The Class of Convex Functions $\mathcal{C}$.

Definition 1.4.1.1

A domain D in $\mathbb{C}$ is said to be convex if the line segment joining any two points of D lies entirely in D.

i.e. $\lambda z_1 + (1-\lambda) z_2 \in D$whenever $z_1, z_2 \in D$ and $0 \leq \lambda \leq 1$. (Goodman, 1983)

Definition 1.4.1.2

A function $f \in \mathcal{A}$ is said to be convex in the open unit disk $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a convex domain. The normalized class of convex univalent functions consists of all functions $f \in S$ for which $f(\mathbb{U})$ is convex and we denote it by $\mathcal{C}$. (see Figure 1.4). (Goodman, 1983)

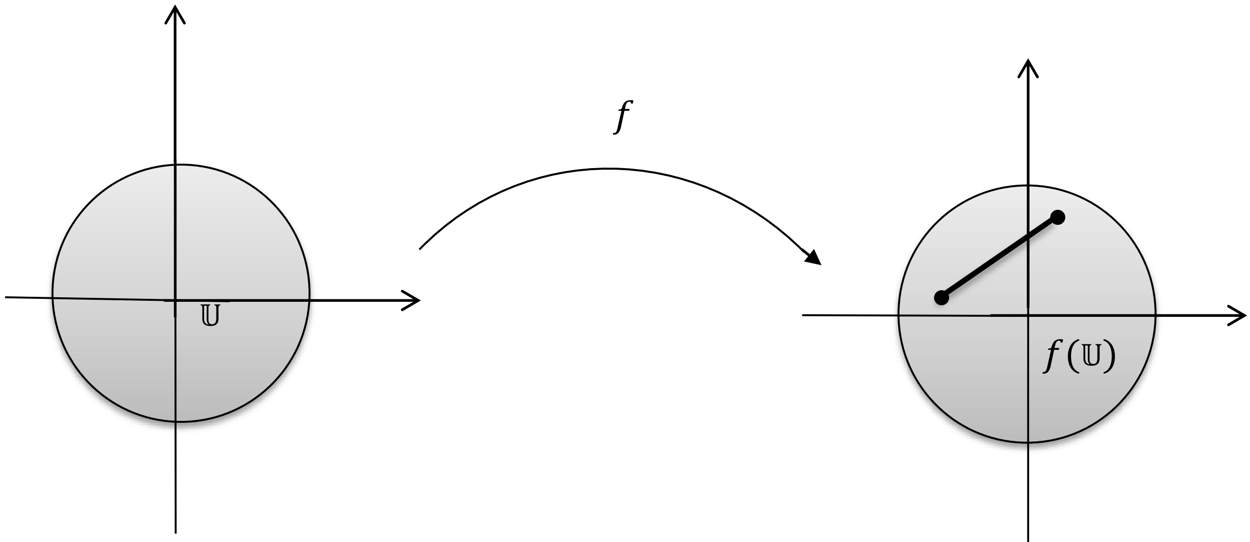


Figure 1.4: The function f maps $\mathbb{U}$ onto a convex domain

Theorem 1.4.1.1

Let $f: \mathbb{U} \to \mathbb{C}$ be a univalent function with $f(0) = 0$ and $f'(0) \neq 0$. Then f is convex if and only if

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0, \ \ (z \in \mathbb{U}). \ \ \text{(Goodman, 1983)}$$

Example 1.4.1.1

The function $f(z) = \frac{z}{1-z}$ is an analytic and univalent function $f(z)$is differentiable in unit disk and satisfies condition $f(0) = 0, f'(0) = \frac{1}{(1-z)^2} = 1.$ Where

$f''(z) = \frac{2(1-z)}{(1-z)^4} = \frac{2}{(1-z)^3}$. Then

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} = \operatorname{Re}\left\{\frac{2z(1-z)^2}{(1-z)^3} + 1\right\} = \operatorname{Re}\left\{\frac{1+z}{1-z}\right\} > 0.$$

Therefore, the function $f(z) = \frac{z}{1-z}$ is convex in $\mathbb{U}$.

Theorem 1.4.1.2" Noshiro-Warschawski Theorem"

In a convex domain D, if f is analytic and $Re\{f'(z)\} > 0$, then f is univalent in D. (Goodman, 1983)

Example 1.4.1.2

If $f(z) = \frac{1+z}{1-z}$ is analytic in a convex domain D and

$$Re\{f'(z)\} = Re\left\{\frac{2}{(1-z)^2}\right\} = Re\left\{\frac{2(1-2xy+x^2-y^2)}{(1-2x+x^2-y^2)^2+4(xy-y)^2}\right\} > 0,$$

then $f(z) = \frac{1+z}{1-z}$ is univalent in D.

1.4.2 The Class of Starlike Functions S^*.

Definition 1.4.2.1

A domain D in $\mathbb{C}$ is said to be starlike with respect to w_0 if each ray from w_0 intersects D in a line segment or a ray. In the special case that $w_0 = 0$ we say that the domain is starlike. (Goodman, 1983)

Definition 1.4.2.2

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike domain. The normalized class of starlike univalent functions with respect to the origin consists of all functions $f \in S$ for which $f(\mathbb{U})$ is starlike and we denote it by S^*. (see Figure 1.5). (Goodman, 1983)

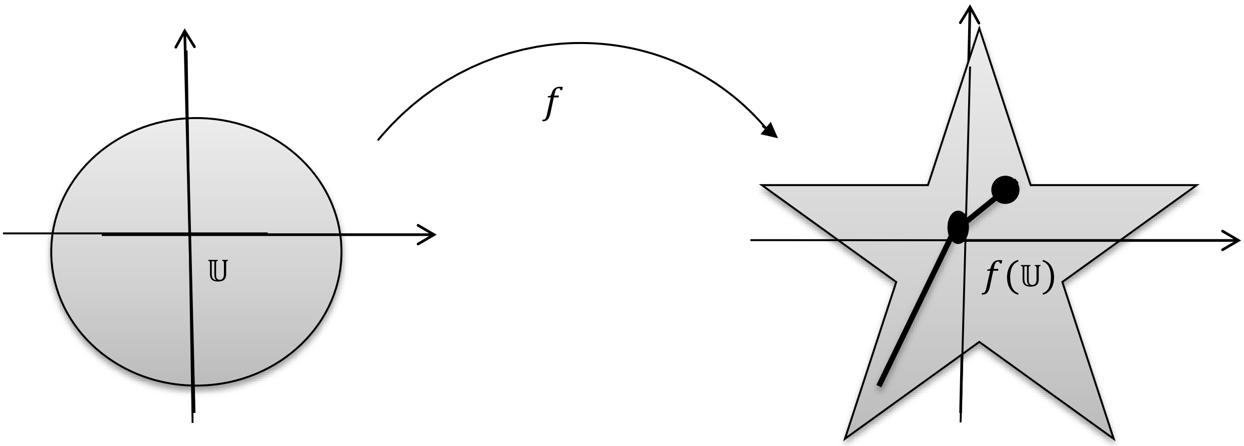


Figure 1.5: The function f maps $\mathbb{U}$ onto starlike domain

Theorem 1.4.2.1

Let $f: \mathbb{U} \to \mathbb{C}$ be univalent function with $f(0) = 0 \; and \; f'(0) \neq 0$. Then f is starlike if and only if

$$\operatorname{Re}\left\{\frac{zf\prime(z)}{f(z)}\right\} > 0, \;\; (z \in \mathbb{U}). \;\; \text{(Goodman, 1983)}$$

Example 1.4.2.1

From Example .1.2.2 the function $\mathcal{K}(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k\, z^k$, for$|z| < 1$, is an analytic function and univalent function. We now prove in the class S^*

$$\operatorname{Re}\left\{\frac{z\mathcal{K}'(z)}{\mathcal{K}(z)}\right\} = \operatorname{Re}\left\{\frac{z(1+z)\cdot(1-z)^2}{(1-z)^3\cdot z}\right\} = \operatorname{Re}\left\{\frac{(1+z)}{(1-z)}\right\} > 0.$$

Therefore, the function $\mathcal{K}(z) = \frac{z}{(1-z)^2}$ is starlike in $\mathbb{U}$.

Example 1.4.2.2

The convex and starlike domains are shown in Figures 1.6 and 1.7,respectively. The domain shown in Figure 1.7 is starlike for w_0 but it is not starlike for the origin.

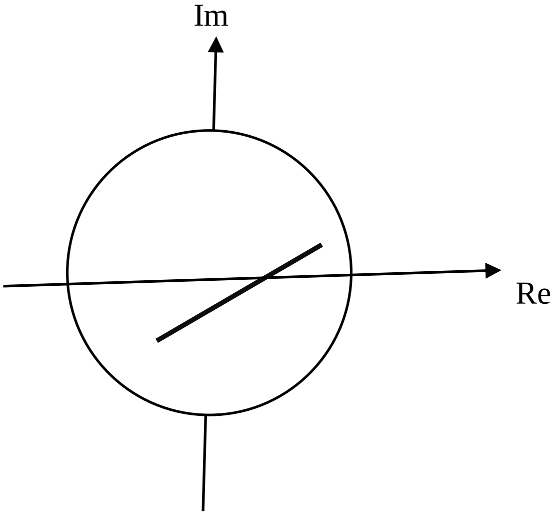


Figure 1.6

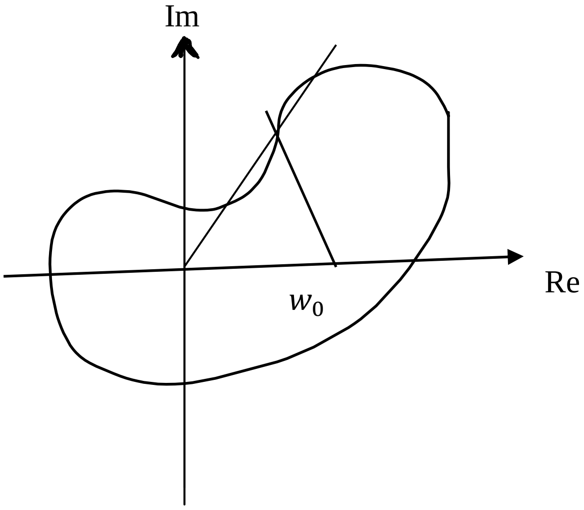


Figure 1.7

Figure 1.6: convex domain

Figure 1.7: starlike domain

Example 1.4.2.3

The M**o**bius function $L_0(z) = \frac{1+z}{1-z}$ is a convex function because it maps $\mathbb{U}$ onto a half-plane.(Thomas et al., 2018)

$$w = L_0(z) = \frac{1+z}{1-z}.$$

Putting $z = x + iy$ and $w = u + iv$.

$$\Rightarrow u + iv = \frac{(1+x) + iy}{(1-x) - iy} = \frac{[(1+x) + iy][(1-x) + iy]}{(1-x)^2 + y^2}$$

$$\Rightarrow u + iv = \frac{1-x^2-y^2+2yi}{(1-x)^2+y^2}.$$

Equating real and imaging parts, we get

$$u = \frac{1-x^2-y^2}{(1-x)^2+y^2}. \qquad \text{(i)}$$

But $|z| < 1$

$$\Rightarrow x^2 + y^2 < 1 \Rightarrow 1 - x^2 - y^2 > 0. \qquad \text{(ii)}$$

From (i) and (ii), we have $u > 0$ That is $Re(w) > 0$.

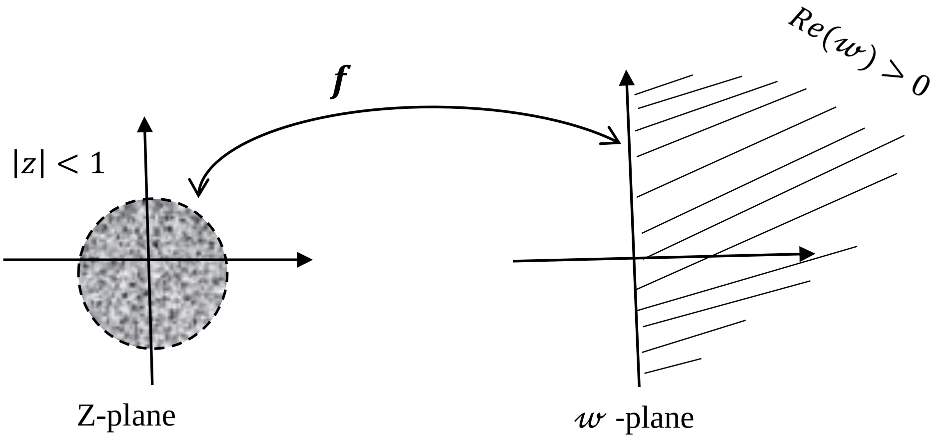


Figure 1.8:The M**o**bius function

Notes:

1. Any circular disk or any half-plane is a convex set. (Goodman, 1983)
2. Intersection of any number of convex sets is a convex set (the intersection may be empty or consist of only one point). (Goodman, 1983)
3. A convex set D is starlike with respect to each interior point of D. Conversely, if a set D is starlike with respect to each interior point of D, then D is a convex set. (Goodman, 1983)

Theorem 1.4.2.2 "Alexander's theorem"

Suppose that f is an analytic function in $\mathbb{U}$ with. Then $f \in C$ if and only if

$zf'(z) \in S^*,\ z \in \mathbb{U}$**.** (Duren 1983)

The Alexander's theorem can be rephrased in the form $f \in S^*$, then the function

$$g(z) = \int_0^z \frac{f(t)}{t} dt,$$

is convex in $\mathbb{U}$.

Example 1.4.2.4

Let the function $f(z) = \frac{z}{1-z}$ is convex then the function

$g(z) = zf'(z)\frac{z|(1-z)-z(-1)|}{(1-z)^2} = \frac{z}{(1-z)^2}$ is starlike in $\mathbb{U}$.

1.4.3 The Class $\mathcal{P}$ of Functions with Positive Real Part.

The class of functions with positive real part plays a crucial role in the Geometric Function Theory. Its significance can be seen from the fact that all the simple subclasses of the class of univalent functions have been defined by using the concept of the class of functions with positive real part. In this section, we define the class of functions with positive real part and we present here some of its interesting properties, such as its relation with the class of univalent functions. In the following, the basic properties of functions with positive real part in the unit disk $\mathbb{U}$ will be given. Also, the concept of subordination in the complex plane will be discussed.

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions.

If g is univalent in $\mathbb{U}$, then it is obvious that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. (Goodman, 1983)

The main property of this class is given by

Definition 1.4.3.1

The set $\mathcal{P}$ is the set of all functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_k z^k + \cdots = 1 + \sum_{k=1}^{\infty} p_k z^k.$$

That is analytic in $\mathbb{U}$, and such that for z in $\mathbb{U}$, $Re(p(z)) > 0$.

Any function in $\mathcal{P}$ is called a function with the positive real part in $\mathbb{U}$. (Goodman, 1983)

Definition 1.4.3.2

The set $\mathcal{P}(\lambda)$ is the set of all functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_k z^k + \cdots = 1 + \sum_{k=1}^{\infty} p_k z^k.$$

That is analytic in $\mathbb{U}$, with the condition $Re(p(z)) > \lambda$ in the unit disk.

Theorem 1.4.3.1

Suppose p is an analytic function in $\mathbb{U}$ satisfying $p(0) = 1$. Then $p \in \mathcal{P}$ if and only if

$p(z) = \int_0^{2\pi} \frac{1+ze^{-i\gamma}}{1-ze^{-i\gamma}} d\rho(\gamma)$ where $d\rho(\gamma) \geq 0$ and $\int_0^{2\pi} d\rho(\gamma) = \rho(2\pi) - \rho(0) = 1$.

(Goodman, 1983)

Lemma 1.4.3.1

Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, then

$$|p_k| \leq 2, \quad k \geq 1.$$

The coefficient bound is sharp for each k. (Goodman, 1983)

Proof

From Theorem 1.4.3.1 and $p \in \mathcal{P}$, we have $p(z) = \int_0^{2\pi} \frac{1+ze^{-i\gamma}}{1-ze^{-i\gamma}} d\rho(\gamma)$where

$d\rho(\gamma) \geq 0$ and $\int_0^{2\pi} d\rho(\gamma) = \rho(2\pi) - \rho(0) = 1$. Therefore

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-i\gamma}}{1 - ze^{-i\gamma}} d\rho(\gamma)$$

$$p(z) = 1 + \sum_{k=1}^{\infty} \left(2\int_0^{2\pi} e^{-ik\gamma} d\rho(\gamma) \right) z^k.$$

Now from $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$yeilde $p_k = 2\int_0^{2\pi} e^{-ik\gamma} d\rho(\gamma)$. Then,

$$|p_k| = 2\left| \int_0^{2\pi} e^{-ik\gamma} d\rho(\gamma) \right|$$

$$\leq 2\int_0^{2\pi} \left| e^{-ik\gamma} \right| |d\rho(\gamma)|$$

$$= 2\int_0^{2\pi} d\rho(\gamma) = 2\big(\rho(2\pi) - \rho(0)\big) = 2(1) = 2.$$

From the Mobius function $L_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{k=1}^{\infty} z^k \Rightarrow |p_k| \leq 2$.

Example 1.4.3.1

1. $f(z) = 1 + z^k$ is in $\mathcal{P}$ for any integer k ≥ 0, but if k ≥ 2, the function $f(z)$ is not univalent.(Goodman, 1983)

2. The Mobius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{k=1}^{\infty} z^k,$$

then this function is in class $\mathcal{P}$, it is analytic and univalent in $\mathbb{U}$.

1.4.4 Close-to-Convex Functions.

Definition 1.4.4.1

A function f analytic in the unit disk is said to be close-to-convex if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0,\ for\ all\ z \in \mathbb{U}.$$ (Goodman, 1983)

We denote $\mathcal{CC}$ the class of close to convex functions in $\mathbb{U}$.

Example 1.4.4.1

In $\mathbb{U}$, the function $f(z) = \frac{z}{1-z}$ is analytic, and choose convex function

$g(z) = \frac{1+z}{1-z}$ satisfies the condition $Re\left\{\frac{f'(z)}{g'(z)}\right\} = Re\left\{\frac{(1-z)^2}{2(1-z)^2}\right\} = Re\left\{\frac{1}{2}\right\} > 0.$

Hence, function $f(z) = \frac{z}{1-z}$ is close-to-convex.

Note:

All starlike functions are close to convex, and all convex functions are close to convex. (Goodman, 1983)

Theorem 1.4.4.1

Every close-to-convex function is univalent. (Duren, 2001)

Proof

If f is close-to-convex in $\mathbb{U}$, then by Definition 1.4.4.1 there exists a convex function $g \in \mathbb{U}$, hence $Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0$. Since $g \in \mathbb{U}$ is convex, then g is univalent function. Hence g^{-1} is exists in convex domain $g(\mathbb{U})$ and consider the function

$$h(w) = f\left(g^{-1}(w)\right); w \in g(\mathbb{U}), \tag{1.2}$$

where g and g^{-1} are analytic functions. Using the fact that the composition of two analytic functions is analytic, the function $h(w)$is an analytic function in $\mathbb{U}$. By differentiating (1.2), we get

$$h'(w) = \frac{f'\left(g^{-1}(w)\right)}{g'\left(g^{-1}(w)\right)} = \frac{f'(z)}{g'(z)},$$

so $Re\{h'(w)\} > 0$ in $g(\mathbb{U})$.by Theorem 1.4.1.2 h is univalent in $g(\mathbb{U})$ thus (1.2) become

$$f = h\left(g(z)\right), z \in \mathbb{U}.$$

Since two univalent functions have a univalent composition, we deduced that f is univalent in $\mathbb{U}$.

Example 1.4.4.2

From Example 1.4.4.1 The function $f(z) = \frac{z}{1-z}$ is close-to-convex and also a univalent function.

1.4.5 Subclasses of Univalent Functions of Order $\boldsymbol{\beta}$.

Definition 1.4.5.1

A function f is said to be convex of order β, $0 \leq \beta < 1$ if

$$\text{Re}\left\{1 + \frac{(zf''(z))}{f'(z)}\right\} > \beta,\ (0 \leq \beta < 1).$$

For $z \in \mathbb{U}$. The class of all convex functions of order β will be denoted by $C(\beta)$.

If $\beta = 0, then\ \ C(\beta) \subset C(0) \equiv C \subset S$. (Goodman, 1983)

Definition 1.4.5.2

A function f is said to be starlike of order $\beta, 0 \leq \beta < 1$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta,\ (0 \leq \beta < 1).$$

For $z \in \mathbb{U}$. The class of all starlike functions of order β will be denoted by $S^*(\beta)$.

If $\beta = 0, then\ S^*(\beta) \subset S^*(0) \equiv S^* \subset S$. (Goodman, 1983)

Definition 1.4.5.3

The subclasses $S^*(\delta)$ and $C(\delta)$ of class $\mathcal{A}$ are defined using the principle of subordination between analytic functions, where δ is an analytic function with postive real part in $\mathbb{U}$, as follows

$$S^*(\delta)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\delta(z)\quad in\ \mathbb{U}\right\},$$

$$\mathcal{C}(\delta)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\delta(z)\quad in\ \mathbb{U}\right\}.$$ (Thomas et al., 2018)

1.5 Coefficients of the Inverse Function.

We can give the formula for the coefficients in the Maclaurin series for

the inverse of a function.

Definition 1.5.1 "Inverse of Function"

Suppose that

$$w=f(z)=z+\sum_{k=2}^{\infty}a_kz^k \text{ and } z=g(w)=w+\sum_{k=2}^{\infty}b_kw^k,$$

be the inverse function, which will always exist, in a unit disk with a center at $w=0$.

Our object is to find $\mathrm{b_k}$ as a function of $a_2,\dots,a_k$.(Goodman, 1983)

Theorem 1.5.1

If $f(z)$ is univalent function and $g(w)$ is inverse function, where

$$w=f(z)=z+\sum_{k=2}^{\infty}a_kz^k,$$

and

$$z=g(w)=w+\sum_{k=2}^{\infty}b_kw^k,$$

then

$$b_k=\frac{1}{k!}\frac{d^{k-1}}{dz^{k-1}}[\hbar(z)]^{-k}\bigg|_{z=0}.$$

Where

$$\hbar(z)=1+\sum_{k=2}^{\infty}a_kz^{k-1}.$$ (Goodman, 1983)

Definition 1.5.2 "bi-univalent function"

A function $f\in\mathcal{A}$ is said to be the bi-univalent function in $\mathbb{U}$. If f and f^{-1} both are univalent in $\mathbb{U}$.(Juma & Aziz, 2012)

Lemma 1.5.1

Fekete and Szego's Theorem states that for $f \in S$and given by (1.1),

$$|a_3 - \mu a_2| \leq 1 + 2\, exp\left(\frac{-2\mu}{1-\mu}\right),$$

$$|a_3 - \mu a_2| \leq \begin{cases} 3 - 4\mu & if \quad \mu \leq 0 \\ 4\mu - 3 & if \quad \mu \geq 1 \end{cases}$$

For $0 \ \leq \ \mu \ \leq \ 1$and the inequality is sharp.(Fekete & Szegö, 1933)

1.6 Some Operators of Analytic Functions.

The study of operators plays a vital role in complex function theory. In the literature, there are several well-known operators such as the Rushewey derivative operator, S$\breve{a}$l$\breve{a}$gean derivative operator, Al-Oboudi derivative operator, and Catas differential operator, which are defined as the following.

Definition 1.6.1

The pochhammer symbol (or the shifted factorial) which is denoted by $(c)_k$ is defined (in terms of the Gamma function) by

$$(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)} = \begin{cases} 1 & for\ k = 0, c \in \mathbb{C} \setminus \{0\} \\ c(c+1)(c+2)\ldots(c+k-1) & for\ k \in \mathbb{N}, c \in \mathbb{C}, \end{cases}$$

where $\Gamma(c)$ is the gamma function is defined for all complex numbers except the non-positive integers.(Norman et al., 2004)

Definition 1.6.2

Let a function f defined on the class $\mathcal{A}$. Then the Rusheweyh derivative operator is defined as the following

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

$$= z + \sum_{k=2}^{\infty} c(n,k) a_k z^k$$

$$= z + \sum_{k=2}^{\infty} \frac{(n+1)_{k-1}}{(k-1)!} a_k z^k,$$

where $(c)_k$ denotes the pochhammer symbol. (Ruscheweyh, 1975)

Definition 1.6.3

Let a function f defined on the class $\mathcal{A}$ and $n \in \mathbb{N}_0$, then the Salagean derivative operator defined as the following

$$S^n f(z) = (z + \sum_{k=2}^{\infty} k^n z^k) * f(z)$$

$= z + \sum_{k=2}^{\infty} k^n a_k z^k,\ (n \in \mathbb{N}_0)$.(Salagean, 1983)

Definition 1.6.4

Let a function f defined on the class $\mathcal{A}$, then, the Al-Oboudi derivative operator defined as the following

$S_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k,\ (n \in \mathbb{N})$.(Al-Oboudi, 2004)

Definition 1.6.5

Let a function f defined on the class $\mathcal{A}$ and $m, \beta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, \lambda \geq 0$, $l \geq 0$. Then the Catas differential $I^m(\lambda, \beta, l)$ operator is defined as the following

$$= z + \sum_{k=2}^{\infty} \left[\frac{1 + \lambda(k-1) + l}{1 + l}\right]^m c(\beta, k) a_k z^k,$$

$= z + \sum_{k=2}^{\infty} \left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m \frac{(\beta+1)_{k-1}}{(k-1)!} a_k z^k$.(Catas & Borsa, 2009)

Chapter Two

Some Applications of Generalized Derivative Operator in the Field of Geometric Function Theory

2.1 Introduction.

The very first work to be devoted exclusively to the subject of fractional calculus was published in 1974. Ever since then, numerous monographs and books as well as scientific research journals have appeared in the existing literature on the theory and applications of fractional calculus. (Srivastava, 2018)

Srivastava et al.(Srivastava, 1989) geometrically explored the class of complex fractional operators (differential and integral) and (Ibrahim, 2011) introduced a generalization for the Srivastava and Owa fractional operators in the unit disk and Srivastava and Owa gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ provided the generality for a class of analytic functions into two-dimensional fractional parameters in $\mathbb{U}$. Where (Khan et al., 2022) used these operations to illustrate a different subclasses of analytical functions .

In this chapter, we use fractional calculus to find some applications of the generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$, where we defined the generalized derivative operator on a new class $S^*(\lambda_1, \lambda_2, l, n, m, \delta)$ of normalized analytic and functions in the open unit disk. .

A function $f \in \mathcal{A}$ is called bounded turning if it satisfies the condition

$$Re\big(f'(z)\big) > 0.$$

The class $\mathcal{T}$ of bounded turning functions, which can be defined as:

$$\mathcal{T} = \left\{f \in \mathcal{A} \; ; \; f'(z) \prec \frac{1+z}{1-z} \; ; \; z \in \mathbb{U}\right\}.$$ (Khan et al., 2022)

To derive the generalized derivative operator(Amer & Darus, 2011), we define the analytic function

$$\phi^m(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \qquad (2.1)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,\ldots.\}$ and $\lambda_2, \lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$.

Now, in (Amer & Darus, 2011) the authors introduced the generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ as the following :

Definition 2.1.1

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n): A \longrightarrow A$

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = \phi^m(\lambda_1,\lambda_2,l)(z) * R^n f(z), (z \in \mathbb{U}), \qquad (2.2)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,\ldots.\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator (Ruscheweyh, 1975), and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, (n \in \mathbb{N}_0, z \in \mathbb{U}),$$

If f is given by (1.1), then we easily find from the equality (2.2) that

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m} c(n,k) a_k z^k,$$

Special cases of this operator include:

- the Ruscheweyh derivative operator (Ruscheweyh, 1975) in the cases:

$$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n) \equiv I^0(0,0,0,n)$$

$$\equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n$$

- the Salagean derivative operator (Salagean, 1983):

$$I^{m+1}(1,0,0,0) \equiv S^n,$$

- The generalized Ruscheweyh derivative operator (Shaqsi & Darus, 2008):

$$I^2(\lambda_1,0,0,n) \equiv R_\lambda^n,$$

- The generalized Salagean derivative operator introduced by Al-Oboudi (Al-Oboudi, 2004):

$$I^{m+1}(\lambda_1, 0,0,0) \equiv S_\beta^n,$$

- The generalized Al-Shaqsi and Darus derivative operator (Al-Shaqsi & Darus, 2008):

$$I^{m+1}(\lambda_1, 0,0, n) \equiv R_{\lambda,\beta}^n,$$

- The Al-Abbadi and Darus generalized derivative operator (Al-Abbadi & Darus, 2009):

$$I^m(\lambda_1, \lambda_2, 0, n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m} \ ,$$

and finally

- The Catas derivative operator (Catas & Borsa, 2009):

$$I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda, \beta, l).$$

Using simple computation one obtains the next result.

$$(1 + l)I^{m+1}(\lambda_1, \lambda_2, l, n)f(z) = (1 + l - \lambda_1)[I^m(\lambda_1, \lambda_2, l, n) * \phi^1(\lambda_1, \lambda_2, l)(z)]f(z)$$

$$+\lambda_1 z[(I^m(\lambda_1, \lambda_2, l, n) * \phi^1(\lambda_1, \lambda_2, l)(z)]'.$$

Where $(z \in \mathbb{U})$ and $\phi^1(\lambda_1, \lambda_2, l)(z)$ an analytic function and form (2.1) given by

$$\phi^1(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1 + \lambda_2(k-1))} z^k.$$

Definition 2.1.2

Let $\delta(z)$ belong to the class S. Then $\delta(z)$ is convex with a positive real part and symmetric about the real axis.

Definition 2.1.3

Let the function f be given by (1.1). Then, the function is in the class $S^*\,(\lambda_1, \lambda_2, l, n, m, \delta)$ if and only if

$$S^*\,(\lambda_1, \lambda_2, l, n, m, \delta) \ = \left\{ f \in \mathcal{A} : \ \frac{z\big(I^m(\lambda_1, \lambda_2, l, n)f(z)\big)'}{I^m(\lambda_1, \lambda_2, l, n)f(z)} \prec \delta(z)\,, \delta(0) = 1 \ \right\}.$$

Definition 2.1.4

Let the function f be given by (1.1). Then, the function is in the class $S^*\left(\lambda_1, \lambda_2, l, n, m, \mathrm{e}^{\lambda z}\right)$ if and only if

$$S^*\left(\lambda_1, \lambda_2, l, n, m, \mathrm{e}^{\lambda z}\right) = \left\{ f \in \mathcal{A} \colon \frac{z\left(I^m(\lambda_1, \lambda_2, l, n) f(z)\right)'}{I^m(\lambda_1, \lambda_2, l, n) f(z)} \prec \mathrm{e}^{\lambda z}, 1 < |\lambda| \leq \frac{\pi}{2} \right\}.$$

Lemma 2.1.1

For $q \epsilon \mathbb{C}$ and a positive integer m, the class of analytic functions is given by

$$\mathcal{G}(f, m) = \{f \colon f(z) = q + q_m z^m + q_{m+1} z^{m+1} + \cdots\}.$$

i. Let $j \in R$. Then

$$Re\big(f(z) + jzf'(z)\big) > 0 \longrightarrow Re\big(f(z)\big) > 0.$$

Moreover,$j > 0$ and $f \epsilon \mathcal{G}(1, m)$, then there is constant $\alpha > 0$ and $\mathscr{w} > 0$ such that

$$\mathscr{w} = \mathscr{w}(j, \alpha, m),$$

and

$$f(z) + jzf'(z) \prec \left(\frac{1+z}{1-z}\right)^{\mathscr{w}} \longrightarrow f(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

ii. For $f \epsilon \mathcal{G}(1, m)$, and for fixed real number $j > 0$ and let $d \in [0,1)$, so that

$$Re\left(f^2(z) + 2f(z)\big(zf'(z)\big)\right) > d \longrightarrow Re\big(f(z)\big) > j.$$

iii. Let $f \epsilon \mathcal{G}(f, m)$, with $Re(f) > 0$, then

$$Re(f(z) + zf'(z) + z^2 f''(z)) > 0,$$

or for $k \colon \mathbb{U} \longrightarrow R$, such that

$$Re\left(f(z) + \left(\frac{zf'(z)}{f(z)}\right) k(z)\right) > 0.$$

Then

$Re\big(f(z)\big) > 0$. (Miller & Mocanu, 2000)

2.2 Problems about Subclasses of Analytic Functions.

Theorem 2.2.1

Let $f\epsilon\mathcal{A}$, and consider the following:

(1) If $I^m(\lambda_1,\lambda_2,l,n)f(z)$ is of bounded turning function, then $\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)$, for some $\lambda \in [0,1)$.

(2) If $\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' \prec \left(\frac{1+z}{1-z}\right)^{w}$, then $\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)$, for some $\lambda \in [0,1)$.

(3) If $Re\left(\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)\right) > \frac{d}{2}, d\epsilon\ [0,1)$, then

$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)$, for some $\lambda \in [0,1)$.

(4) If $Re\left(z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'' \_\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + 2\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)\right) > 0$,

then $\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)$, for some $\lambda \in [0,1)$.

(5) If $Re\left(\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) + 2\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)\right) > 1$, then

$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)$, for some $\lambda \in [0,1)$.

Proof

(1) Define a function $p(z)$ as follows:

$$p(z) = \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}, z \in \mathbb{U}.$$

$$\Rightarrow zp(z) = I^m(\lambda_1,\lambda_2,l,n)f(z)$$

$$\Rightarrow zp'(z) + p(z) = [I^m(\lambda_1,\lambda_2,l,n)f(z)]'$$

$$= 1 + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}\ k.c(n,k)a_k z^{k-1}.$$

From (1), we have $I^m(\lambda_1,\lambda_2,l,n)f(z)$ is bounded turning function, and this gives

$$\Rightarrow Re[zp'(z) + p(z)] > 0$$

$$\Rightarrow Re[\, I^m(\lambda_1,\lambda_2,l,n)f(z)]' > 0,$$

and from Lemma 2.1.1, part (i)

$$Re\left(p(z)\right) > 0 \Rightarrow p(z) = \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}(\lambda)\,.$$

(2) Suppose that

$$P(\mathrm{z}) = \frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z}, z \in \mathbb{U}.$$

$$\Rightarrow zp(z) = I^m(\lambda_1, \lambda_2, l, n)f(z)$$

$$\Rightarrow zp'(z) + p(z) = [I^m(\lambda_1, \lambda_2, l, n)f(z)]' \Rightarrow Re[\, I^m(\lambda_1, \lambda_2, l, n)f(z)]' > 0,$$

from Lemma 2.1.1 part (i) we fixed the real number $j > 0$ such that $\mathcal{w} = \mathcal{w}(j)$ and:

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\mathcal{w}},$$

$$\Rightarrow p(z) = \frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{j}$$

$$\Rightarrow \; Re\left[\frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z}\right] > j > 0$$

$$\Rightarrow \; \frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z} \in \mathcal{P}$$

$$\therefore \; \frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z} \in \mathcal{P}.$$

(3) From Lemma 2.1.1, part (ii). Suppose that:

$Re\,\left(p^2(z) + 2p(z).z\,p'(z)\right)$

$$= Re\left[\frac{I^{m2}(\lambda_1, \lambda_2, l, n)f(z)}{z^2} + 2\,\frac{I^m(\lambda_1, \lambda_2, l, n)f(z)}{z}\right.$$

$$\left.\cdot\, z\left(\frac{z\big(I^m(\lambda_1, \lambda_2, l, n)f(z)\big)' - I^m(\lambda_1, \lambda_2, l, n)f(z)}{z^2}\right)\right]$$

$$= Re\left[\frac{I^{m^2}(\lambda_1, \lambda_2, l, n)f(z)}{z^2} + \frac{2I^m(\lambda_1, \lambda_2, l, n)f(z)[I^m(\lambda_1, \lambda_2, l, n)f(z)]'}{z}\right.$$

$$\left.- \frac{2I^{m^2}(\lambda_1, \lambda_2, l, n)f(z)}{z^2}\right]$$

$$= Re\left[\frac{2I^m(\lambda_1, \lambda_2, l, n)f(z)[I^m(\lambda_1, \lambda_2, l, n)f(z)]'}{z} - \frac{I^{m^2}(\lambda_1, \lambda_2, l, n)f(z)}{z^2}\right]$$

$$= 2Re\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{z} - \frac{I^{m^2}(\lambda_1,\lambda_2,l,n)f(z)}{2z^2}\right]$$

$$= 2Re\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\left(\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{2z}\right)\right] > d$$

$$= Re\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\left(\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{2z}\right)\right] > \frac{d}{2}.$$

From Lemma 2.1.1 part (ii) for fixed real number $j > 0$ and satisfying the condition

$$\Rightarrow Re\ \left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right] > j > 0,$$

$$\therefore \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}.$$

(4) Suppose that

$$Re\ (p(z) + zp'(z) + z^2p''(z))$$

$$= Re\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + z\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - I^m(\lambda_1,\lambda_2,l,n)f(z)}{z^2}\right)\right.$$

$$+ \frac{z^2\Big(z^2\Big(\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)' + z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'' - \big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'\Big)\Big)}{z^4}$$

$$\left. - \frac{z^2\Big(2z\Big(z\cdot\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)' - I^m(\lambda_1,\lambda_2,l,n)f(z)\Big)\Big)}{z^4}\right],$$

$$Re\ (p(z) + zp'(z) + z^2p''(z))$$

$$= Re\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + \left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'\right.$$

$$- \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''$$

$$\left. - 2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \frac{2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)}{z}\right],$$

$Re\ (p(z) + zp'(z) + z^2 p''(z))$

$$= Re\left[z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'' - \left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \frac{2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)}{z}\right].$$

From Lemma 2.1.1 part (iii) implies

$$\Rightarrow Re\ \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) > 0$$

$$\therefore\ \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}\ .$$

(5) Suppose that

$$\mathrm{Re}\left(p(z) + \frac{zp'(z)}{p(z)} + z^2 p''(z)\right)$$

$$= \mathrm{Re}\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + \left(\frac{z\left[\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - I^m(\lambda_1,\lambda_2,l,n)f(z)}{z^2}\right]}{\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}}\right)\right.$$

$$+ \frac{z^2\left(z\left(\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)\right)'' + \left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - \left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'\right)}{z^2}$$

$$\left. - \frac{\left(z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' - I^m(\lambda_1,\lambda_2,l,n)f(z)\right).2z}{z^2}\right]$$

$$= \mathrm{R}e\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1 + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''\right.$$

$$\left. - \frac{2z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{z} + \frac{2I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right).$$

From Lemma 2.1.1, part (iii) where $k(z) = 1$.

$$Re\big(p(z)\big) = Re\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) > 0$$

$$\therefore \ \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \in \mathcal{P}. \ \blacksquare$$

To establish the upper bounds of the operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$, we employ the exponential integral within the unit disk.

Theorem 2.2.2

Let $f \in S^*(\lambda_1,\lambda_2,l,n,m,\delta)$, where $\delta(z)$ is convex in $\mathbb{U}$. Then,

$$I^m(\lambda_1,\lambda_2,l,n)f(z) \prec z\, exp \int_0^z \frac{\delta\big(w(\xi)\big)-1}{\xi}\, d\xi, \tag{2.3}$$

where $w(z)$ is analytic in $\mathbb{U}$ having condition $w(0)=0$ and $|w(z)| < 1$. Furthermore, for $|z| = \xi$, we have

$$\exp \int_0^1 \frac{\delta\big(w(-\xi)\big)-1}{\xi}\, d\xi \leq \left| \frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \right| \leq \exp \int_0^1 \frac{\delta\big(w(\xi)\big)-1}{\xi}\, d\xi.$$

Proof.

From Definition 2.1.3 we get

$$\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \delta(z),$$

from the definition of subordination, we get

$$\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} = \delta\big(w(z)\big), z \in \mathbb{U}$$

$$\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1 = \delta\big(w(z)\big) - 1$$

$$z\left[\frac{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - \frac{1}{z}\right] = \delta\big(w(z)\big) - 1\,,$$

dividing by z

$$\frac{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - \frac{1}{z} = \frac{\delta\big(w(z)\big)-1}{z}$$

$$\int_0^z \frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\,dz-\int_0^z \frac{1}{z}\,\mathrm{dz}\;=\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi$$

$$log\,I^m(\lambda_1,\lambda_2,l,n)f(z)-log\,z=\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi$$

$$log\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\;=\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi \qquad (2.4)$$

$$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}=exp\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi$$

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=z\,exp\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi$$

$$\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=\frac{1}{z}+\frac{\delta\left(w(z)\right)-1}{z}$$

$$\frac{\mathrm{z}\left(I^m(\lambda_1,\lambda_2,l,n)f(\mathrm{z})\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=1+\delta\left(w(z)\right)-1,$$

we obtain

$$I^m(\lambda_1,\lambda_2,l,n)f(z)\prec z\,exp\int_0^z \frac{\delta\left(w(\xi)\right)-1}{\xi}\,d\xi.$$

Hence (2.3) is proved.

From Definition 2.1.2, the function $\delta(z)$ is convex and symmetric for the real axis, where $0<|z|<\xi<1$. That is

$$\delta(-\xi\,|z|)\le \mathrm{Re}\{\delta\left(w(\xi z)\right)\}\le\delta(\xi\,|z|),\qquad (0<\xi\;<1\;,z\in\mathbb{U}\,).$$

Then we have the inequalities

$$-|z|>-1\qquad\quad\rightarrow\;|z|<1$$

$$\delta(-\xi\,)\le\delta(-\xi\,|z|),\delta(\xi\,|z|)\le\delta(\xi).$$

Consequently, we get

$$\delta(w(-\xi\,|z|)) \le \operatorname{Re}\{\delta(w(\xi))\} \le \delta(w(\xi\,|z|))$$

$$\delta(w(-\xi\,|z|)) - 1 \le \operatorname{Re}\{\delta(w(\xi))\} - 1 \le \delta(w(\xi\,|z|)) - 1$$

$$\frac{\delta(w(-\xi\,|z|)) - 1}{\xi} \le \frac{\operatorname{Re}\{\delta(w(\xi))\} - 1}{\xi} \le \frac{\delta(w(\xi\,|z|)) - 1}{\xi}$$

$$\int_0^1 \frac{\delta(w(-\xi\,|z|)) - 1}{\xi} d\xi \ \le \int_0^1 \frac{\operatorname{Re}\{\,\delta(w(\xi))\} - 1}{\xi} d\xi \ \le \int_0^1 \frac{\delta(w(\xi\,|z|)) - 1}{\xi} d\xi,$$

from (2.4), we obtain

$$\int_0^1 \frac{\delta(w(-\xi\,|z|)) - 1}{\xi} d\xi \ \le log \left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right| \le \int_0^1 \frac{\delta(w(\xi\,|z|)) - 1}{\xi} d\xi$$

$$exp \int_0^1 \frac{\delta(w(-\xi\,|z|)) - 1}{\xi} d\xi \ \le \left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right| \le exp \int_0^1 \frac{\delta(w(\xi\,|z|)) - 1}{\xi} d\xi,$$

hence, we have

$$exp \int_0^1 \frac{\delta(w(-\xi\,)) - 1}{\xi} d\xi \ \le \left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right| \le exp \int_0^1 \frac{\delta(w(\xi\,)) - 1}{\xi} d\xi\,. \ \blacksquare$$

If δ is convex univalent and $\delta(0) = 1$, then we find a condition on f to be in the class $S^*\,(\lambda_1,\lambda_2,l,n,m,\delta)$.

Theorem 2.1.3

If $f \in \mathcal{A}$ satisfies the subordination condition

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left[2 + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right] \prec \delta(z),$$

Then,

$$f \in S^*\,(\lambda_1,\lambda_2,l,n,m,\delta).$$

Proof.

Let

$$p(z) = \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)},$$

and let $p(z) = 1$ and from Lemma 2.1.1 , part (i)

$$p(z) + p(z)z\big(p(z)\big)' \prec \delta(z)$$

$$\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}$$

$$+\,z\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)\left[z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'' + \big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'\right]}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)^2}\right]$$

$$-\,z\left[\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)^2}\right] \prec \delta(z),$$

then

$$\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)} + \frac{z^2\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)''}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)} + \frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)}$$

$$-\frac{z^2\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'^2}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)^2}$$

$$= \frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left[2 + \frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)''}{I^m(\lambda_1,\lambda_2,l,n)f(z)'}\right.$$

$$\left.-\frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right] \prec \ \delta(z),$$

this implies

$$\operatorname{Re} p(z) > 0 \ \Rightarrow \ p(z) = \frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \ \delta(z),$$

that is

$$f \in S^*\,(\lambda_1,\lambda_2,l,n,m,\delta). \ \blacksquare$$

Theorem 2.2.4

If $f \ \epsilon \mathcal{A}$ and satisfies the inequality

$$1 + \frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)''}{\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'} \prec \mathrm{e}^{\lambda z}\,.$$

Then,

$$f \ \epsilon\, S^*\left(\lambda_1,\lambda_2,l,n,m,\mathrm{e}^{\lambda z}\right).$$

Proof

Let

$$p(z)= \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}$$

$$log\, p(z)=log\, z\left(I^m f(z)\right)'-log\, I^m f(z)$$

$$\frac{p(z)'}{p(z)}=\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}+\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}-\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}$$

$$\frac{zp(z)'}{p(z)}=\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}+1-\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}$$

$$p(z)+\frac{zp(z)'}{p(z)}=\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}+1.$$

Now

$$p(z)+\frac{zp(z)'}{p(z)}=1+\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} \prec \mathrm{e}^{\lambda z},$$

this implies that ((Miller & Mocanu, 2000)p .123)

$$p(z)= \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \mathrm{e}^{\lambda z},$$

then

$$f(z)\ \epsilon\ S^*\left(\lambda_1,\lambda_2,l,n,m,\mathrm{e}^{\lambda z}\right). \ \blacksquare$$

Example 2.2.1

Let

$$\frac{z\,f'(z)}{f(z)}=\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}$$

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=\frac{z}{(1-z)^2}, \qquad f(z)\in\mathcal{A}.$$

Then the solution of $\frac{z\,f'(z)}{f(z)} = \frac{1+z}{1-z}$ is formulated as follows:

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = \frac{z}{(1-z)^2}, f(z) \in \mathcal{A},$$

we have $I^m(\lambda_1, \lambda_2, l, n)f(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k$

$$\frac{z\big(I^m(\lambda_1, \lambda_2, l, n)f(z)\big)'}{I^m(\lambda_1, \lambda_2, l, n)f(z)} = \frac{\sum_{k=1}^{\infty} k^2 z^k}{\sum_{k=1}^{\infty} kz^k}$$

$$\frac{z\big(I^m(\lambda_1, \lambda_2, l, n)f(z)\big)'}{I^m(\lambda_1, \lambda_2, l, n)f(z)} = 1 + 2z + 2z^2 + 2z^3 + \cdots$$

$$\therefore\ \frac{z\big(I^m(\lambda_1, \lambda_2, l, n)f(z)\big)'}{I^m(\lambda_1, \lambda_2, l, n)f(z)} = 1 + 2\sum_{k=1}^{\infty} z^k = \frac{1+z}{1-z}.$$

Moreover, the solution of the equation

$$f(z) + \frac{z\,f'(z)}{f(z)} = \frac{1+z}{1-z},$$

is approximated to

$$f(z) = \frac{z}{1-z}.$$

We have

$$f(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k$$

$$f(z) + \frac{z\,f'(z)}{f(z)} = \sum_{k=1}^{\infty} z^k + \frac{z\left[\sum_{k=1}^{\infty} kz^{k-1}\right]}{\sum_{k=1}^{\infty} z^k} = \sum_{k=1}^{\infty} z^k + \frac{\sum_{k=1}^{\infty} kz^k}{\sum_{k=1}^{\infty} z^k}$$

$$= \sum_{k=1}^{\infty} z^k + 1 + \sum_{k=1}^{\infty} z^k$$

$$\therefore\ f(z) + \frac{z\,f'(z)}{f(z)} = 1 + 2\sum_{k=1}^{\infty} z^k = \frac{1+z}{1-z}.$$

2.3 Applications of Generalized Derivative Operator.

The solution of the complex Briot-Bouquet (BB) differential equation is established in (Miller & Mocanu, 2000). We produce a presentation of our results in complex BB differential equations, and the class of BB differential equations is a link of differential equations whose consequences are visible in the complex plane. The study of new special functions as follows (Khan et al., 2023)

$$\omega\, f(z) + (1-\omega)\frac{zf'(z)}{f(z)} = \varphi(z), \qquad (2.5)$$

$$\varphi(0) = f(0), \qquad \omega \in [0,1].$$

In (Miller & Mocanu, 2000), many new applications of these equations in Geometric Function Theory have been discussed. Now, we investigate (2.5) by using the operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ and find its solutions by applying the subordination relations. The operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ propagates the complex BB differential equation as follows:

$$\omega I^m(\lambda_1,\lambda_2,l,n)f(z) + (1-\omega)\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) = \varphi(z), \qquad (2.6)$$

where,

$$\varphi(0) = f(0)\ ,\omega \in \mathbb{U}.$$

A trivial solution of (2.6) is given when $\omega = 1$; our investigation concerns the case with $f \in \mathcal{A}$ and $\omega = 0$. As follows

Theorem 2.3.1

Let us have equation (2.6) with $\omega = 0$. If $\delta(z)$ is convex in $\mathbb{U}$.Then

$$I^m(\lambda_1,\lambda_2,l,n)f(z) \prec z\, exp\int_0^z \frac{\delta\big(w(\zeta)\big)-1}{\zeta}\, d\zeta \qquad (2.7)$$

where $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$ and $|w(z)| < 1$.

Proof

From equation (2.6), and $f(z) \in \mathcal{A}$. Then, we get

$$Re\left(\frac{z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) > 0,$$

$$Re\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right)>0 \Leftrightarrow Re\left(\frac{z\left(z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_k z^k\right)'}{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_k z^k}\right)>0,$$

$$\Leftrightarrow Re\left(\frac{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}kc(n,k)a_k z^{k-1}\right)}{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_k z^k}\right)>0$$

$$\Leftrightarrow Re\left(\frac{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}kc(n,k)a_k z^{k-1}\right)}{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_k z^{k-1}\right)}\right)>0$$

$$\Leftrightarrow Re\left(\frac{1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}kc(n,k)a_k}{1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_k}\right)>0 \qquad z\longrightarrow 1^+$$

$$\Leftrightarrow Re\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m}kc(n,k)a_k\right)>0.$$

Moreover, by the definition of $I^m(\lambda_1,\lambda_2,l,n)f(z)$, we indicate that

$I^m(\lambda_1,\lambda_2,l,n)f(0)=0.$

Consequently,

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\in\mathcal{P} \quad \Rightarrow f(z)\in S^*\,(\lambda_1,\lambda_2,l,n,m,\delta).$$

Hence, in the light of Theorem 2.2.2, the result given in (2,7) is completed.

Chapter Three

Application of the Generalized Derivative Operator on Analytic Functions to Determine Fekete- Szego Inequalities

3.1 Introduction.

The Fekete-szego inequality is an inequality for the coefficients of univalent analytic functions found by (Fekete & Szegö, 1933), related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete-Szego problem. For the subclass of S, consisting of convex functions C, starlike functions S^*, and close-to-convex functions CC. Various authors have studied sharp upper bounds for the functional $|a_3 - \mu a_2^2|$, however, we mention only a few names here such as (Keogh & Merkes, 1969)**,**(Koepf, 1987), (Ibrahim & Darus, 2001),(Darus, 2000; Darus & Thomas, 1996),(Ibrahim & Darus, 2001). In particular, for $f \in CC$ and be given by (1.1) (Keogh & Merkes, 1969) showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \mu \leq \frac{1}{3} \\ \frac{1}{3} + \frac{4}{9\mu}, & \frac{1}{3} \leq \mu \leq \frac{2}{3} \\ 1, & , \frac{2}{3} \leq \mu \leq 1 \\ 4\mu - 3, & \mu \geq 1 \end{cases}$$

and for each $\mu,$ there is a function in CC for which equality holds.

The class used in this chapter is expanded from the class (Ramesha et al., 1995)**.** Where if the operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ is replaced by the function $f(z)$

in the class used we get the class defined by (Alharayzeh & Darus, 2022)

In this chapter, we obtain results the Fekete-Szego inequality for function in the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z), \psi(z); \alpha, \beta, \gamma)$.

Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$,$\varphi(z) = z + \sum_{k=2}^{\infty} w_k z^k$and $\psi(z) = z + \sum_{k=2}^{\infty} d_k z^k$ be Analytic functions in $\mathbb{U}$ where $b_k,\ w_k, d_k > 0$ and $w_k >\ d_k,$ and we define the Hadamard product as follows:

$$\left.\begin{aligned} g(z) * \varphi(z) &= z + \sum_{k=2}^{\infty} b_k w_k z^k, \\ g(z) * \psi(z) &= z + \sum_{k=2}^{\infty} b_k d_k z^k. \end{aligned}\right\} \tag{3.1}$$

Now we define the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z), \psi(z); \alpha, \beta, \gamma)$ as follows.

Definition 3.1.1

Let the function f be given by (1.1). Then, the function

$f \in \Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z), \psi(z); \alpha, \beta, \gamma); n, m \in \mathbb{N}_0 = \{0,1,2, \ldots .\}, \lambda_2 \geq \lambda_1 \geq 0, l \geq 0, \quad 0 \leq \alpha < 1, 0 \leq \beta < 1,$ and $0 \leq \gamma < 1$ if and only if there exists $g \in \mathcal{A}$,$g(z) \neq 0$ such that:

$$Re\left(\frac{\alpha z^2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{g(z)} + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{g(z)}\right) > \beta, \tag{3.2}$$

$$Re\left(\frac{g(z)*\varphi(z)}{g(z)*\psi(z)}\right) > \gamma, \text{ for } z \in \mathbb{U} \tag{3.3}$$

for some $\varphi(z)$ and $\psi(z)$, both are analytic in $\mathbb{U}$ such that $g(z) * \psi(z) \neq 0$,

$w_k, d_k > 0$ and $w_k > d_k, k \geq 2$.

We use the generalized derivative operator in this work as defined in Definition 2.1.1.

3.2 Problem with Fekete-Szego Inequality.

Lemma 3.2.1

Let h be analytic in $\mathbb{U}$ with $Re\, h(z) > 0$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, for $z \in \mathbb{U}$, then,

$$|c_n| \leq 2, \text{ where } n \geq 1, \left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}. \quad \text{(Pommerenke, 1975)} \tag{3.4}$$

Lemma 3.2.2

Let $g \in S^*$, the starlike function with $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ .Then for μ real,

$$|b_3 - \mu b_2^2| \leq max\{1, |3 - 4\mu|\}. \tag{3.5}$$

(Keogh & Merkes, 1969)

The first result for the class is as follows.

Theorem 3.2.1

Let the function f given by (1.1) belong to the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$ and

$0 \le \alpha < 1$. Then,

$$\left.\begin{aligned} (\alpha+1)\mathrm{A}|a_2| &\le \frac{(w_2-d_2)(1-\ \beta)+1-\gamma}{w_2-d_2}\ , \\ 3(2\alpha+1)\mathrm{B}|a_3| &\le \frac{4(1-\gamma)^2}{(w_3-d_3)(w_2-d_2)}+\frac{4(1-\gamma)(1-\ \beta)}{w_2-d_2}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\ \beta) \end{aligned}\right\} \tag{3.6}$$

Where

$\mathrm{A}=\frac{(1+\lambda_1+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2)^m}c(n,2),\ \ \mathrm{B}=\frac{(1+\lambda_1(2)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(2)\right)^m}c(n,3).$

Proof

From the definition, we have

$$g(z)*\varphi(z)=(p(z)(1-\gamma)+\gamma)\big(g(z)*\psi(z)\big). \tag{3.7}$$

For any $z\in\mathbb{U}$, with $Rep(z)>0$ given by $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$, where

$p_1,p_2,p_3,\cdots\in\mathbb{C}$.

From (3.7), we have

$$z+b_2w_2z^2+b_3w_3z^3+\cdots=(z+b_2d_2z^2+b_3d_3z^3+\cdots)+(p_1(1-\gamma)z^2+ p_1(1-\gamma)b_2d_2z^3+\cdots)+(p_2(1-\gamma)z^3+p_2(1-\gamma)b_2d_2z^4+\cdots). \tag{3.8}$$

Now, equating coefficients, we get

$$b_2(w_2-d_2)=p_1(1-\gamma), \tag{3.9}$$

$$b_3(w_3-d_3)=b_2d_2p_1(1-\gamma)+p_2(1-\gamma). \tag{3.10}$$

And also follows from (3.2) that

$$\alpha z^2\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)''+z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'=\ g(z)(h(z)(1-\beta)+ \beta), \tag{3.11}$$

where $Reh(z)>0$,and writing $h(z)=1+c_1z+c_2z^2+c_3z^3+\cdots,$ where

$c_1,c_2,c_3,\cdots\in\mathbb{C}$, and now

$2\alpha \mathrm{A} a_2 z^2 + 6\alpha \mathrm{B} a_3 z^3 + \cdots + z + 2\mathrm{A} a_2 z^2 + 3\mathrm{B} a_3 z^3 + \cdots = (z + c_1(1-\beta)z^2 + c_2(1-\beta)z^3 + \cdots) + (b_2 z^2 + c_1(1-\beta)b_2 z^3 + \cdots) + (b_3 z^3 + c_1(1-\beta)b_3 z^4 + \cdots) + \cdots,$ (3.12)

and equating coefficients give

$$2(\alpha+1)\mathrm{A} a_2 z^2 = c_1(1-\beta)z^2 + b_2 z^2 \tag{3.13}$$

$$3(2\alpha+1)\mathrm{B} a_3 z^3 = (1-\beta)(c_2 + b_2 c_1)z^3 + b_3 z^3, \tag{3.14}$$

from equation (3.13), (3.14) we obtain

$$2(\alpha+1)\mathrm{A} a_2 = c_1(1-\beta) + b_2 \tag{3.15}$$

$$3(2\alpha+1)\mathrm{B} a_3 = (1-\beta)(c_2 + b_2 c_1) + b_3, \tag{3.16}$$

from equation (3.15) we find a_2^2,

$$\mathrm{A} a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)} + \frac{b_2}{2(\alpha+1)}.$$

We get:

$$a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}} \Rightarrow a_2^2 = \left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2.$$

The result follows applying inequalities:

$|p_1| \le 2, |p_2| \le 2, |c_1| \le 2, |c_2| \le 2$, and from (3.9), (3.10) we get

$b_2 = \frac{2(1-\gamma)}{(w_2-d_2)} \Rightarrow |b_2| \le \frac{2(1-\gamma)}{(w_2-d_2)}$, and $b_3 = \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}$

$$\Rightarrow |b_3| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}.$$

From (3.15) we obtain

$$(\alpha+1)\mathrm{A} a_2 = \frac{c_1(1-\beta)}{2} + \frac{b_2}{2}$$

$$\le \frac{2(1-\beta)}{2} + \frac{2(1-\gamma)}{2(w_2-d_2)}$$

$$\therefore (\alpha+1)\mathrm{A}|a_2| \le \frac{(w_2-d_2)(1-\beta) + 1 - \gamma}{w_2-d_2}.$$

From (3.16) we obtain

$$3(2\alpha+1)\mathrm{B}|a_3| \le 2(1-\beta) + 2(1-\beta)\left(\frac{2(1-\gamma)}{2(w_2-d_2)}\right) + \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3},$$

$$3(2\alpha+1)\mathrm{B}|a_3| \le 2(1-\beta) + \frac{4(1-\beta)(1-\gamma)}{w_2-d_2} + \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}.$$

Now we display the main result for the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$.

Theorem 3.2.2

Let the function f be given by (1.1) and belong to the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$.

Then

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\le \begin{cases} \dfrac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \dfrac{4(1-\gamma)(1-\beta)}{w_2-d_2} + \dfrac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) - \dfrac{3(2\alpha+1)\mathrm{B}\mu\big(1-\gamma+(1-\beta)(w_2-d_2)\big)^2}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} & if\ \mu \le \mu_0\,, \\ \dfrac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} - \dfrac{4(1-\gamma)^2}{(w_2-d_2)^2} + \dfrac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) + \dfrac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\,\mathrm{B}} & if\ \mu_0 \le \mu \le \mu_1, \\ \dfrac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} - \dfrac{2(1-\gamma)^2}{(w_2-d_2)^2} + \dfrac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) & if\ \ \mu_1 \le \mu \le \mu_2\,, \\ -\dfrac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} - \dfrac{4(1-\gamma)(1-\beta)}{w_2-d_2} - \dfrac{2(1-\gamma)}{w_3-d_3} - 2(1-\beta) + \dfrac{3(2\alpha+1)\mathrm{B}\mu\big(1-\gamma+(1-\beta)(w_2-d_2)\big)^2}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} & if\ \mu_2 \le \mu\,, \end{cases}$$

Where

$$\mu_0 = \frac{\mu_1(1-\gamma)}{1-\gamma+(1-\beta)(w_2-d_2)},$$

$$\mu_1 = \frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}},$$

$$\mu_2 = \frac{\mu_0(w_2-d_2)^2}{1-\gamma+(1-\beta)(w_2-d_2)}\left(\frac{4(1-\gamma)d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2}{w_3-d_3} + \frac{2(1-\beta)}{(1-\gamma)} + \frac{2(1-\beta)}{(w_2-d_2)} - \frac{(1-\gamma)}{(w_2-d_2)^2}\right).$$

Proof

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2) = 3(2\alpha+1)\mathrm{B}a_3 - 3(2\alpha+1)\mathrm{B}\mu a_2^2\,. \qquad (3.17)$$

From (3.15) we get, $a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}} \Rightarrow a_2^2 = \left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2,$

from (3.16) and the value a_2^2 substituting by together in (3.17),

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2) = (1-\beta)(c_2+b_2c_1) + b_3 - 3(2\alpha+1)\mathrm{B}\mu\left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2$$

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2)$$

$$= (1-\beta)c_2+(1-\beta)b_2c_1$$

$$+\, b_3 - 3(2\alpha+1)\mathrm{B}\mu\left(\frac{{c_1}^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}+\frac{c_1b_2(1-\beta)}{2(\alpha+1)^2\mathrm{A}^2}+\frac{{b_2}^2}{4(\alpha+1)^2\mathrm{A}^2}\right)$$

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2)=(1-\beta)c_2+(1-\beta)b_2c_1+b_3-\frac{3(2\alpha+1)\mathrm{B}\mu {c_1}^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}-\frac{3(2\alpha+1)\mathrm{B}\mu c_1b_2(1-\beta)}{2(\alpha+1)^2\mathrm{A}^2}-\frac{3(2\alpha+1)\mathrm{B}\mu {b_2}^2}{4(\alpha+1)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2)=b_3-\frac{3(2\alpha+1)\mathrm{B}\mu {b_2}^2}{4(\alpha+1)^2\mathrm{A}^2}+(1-\beta)c_2+b_2c_1(1-\beta)\left(1-\frac{3(2\alpha+1)\mathrm{B}\mu}{2(\alpha+1)^2\mathrm{A}^2}\right)+{c_1}^2(1-\beta)^2\left(\frac{2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mathrm{B}\mu}{4(\alpha+1)^2\mathrm{A}^2}-\frac{1}{2}\right). \tag{3.18}$$

From (3.18), we have,

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\le\left|b_3-\frac{3(2\alpha+1)\mathrm{B}\mu b_2^2}{4(\alpha+1)^2\mathrm{A}^2}\right|+\left|(1-\beta)c_2-\frac{1}{2}c_1^2(1-\beta)^2\right|+\frac{|c_1|^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}|2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}|+\frac{|b_2||(1-\beta)c_1|}{2(\alpha+1)^2\mathrm{A}^2}|2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}|. \tag{3.19}$$

Now, consider the first case for all

$$2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}\ge 0 \implies \therefore\ \mu\le\frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}.$$

Note that

$2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\,\mathrm{B}>0$ and $b_3-\frac{3(2\alpha+1)\mu b_2^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2}>0$, from Lemma 3.2.1

$$\left|c_2(1-\beta)-\frac{c_1^2(1-\beta)^2}{2}\right|\le 2(1-\beta)-\frac{|c_1|^2(1-\beta)^2}{2},$$

and inequalities

$$|b_2|\le\frac{2(1-\gamma)}{(w_2-d_2)}\,,$$

and

$$|b_3| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3},$$

Substituting $|b_2|, |b_3|$ in (3.19)

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} - \frac{(4)3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{4(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}$$

$$+ 2(1-\beta) - \frac{|c_1|^2(1-\beta)^2}{2}$$

$$+ \frac{2(1-\gamma)|c_1|(1-\beta)}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}[2(\alpha+1)^2\mathrm{A}^2 - 3(2\alpha+1)\mu\mathrm{B}]$$

$$+ \frac{|c_1|^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}[2(\alpha+1)^2\mathrm{A}^2 - 3(2\alpha+1)\mu\mathrm{B}]$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} - \frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} + 2(1-\beta) -$$
$$\frac{|c_1|^2(1-\beta)^2}{2} + \frac{2(1-\gamma)|c_1|(1-\beta)2(\alpha+1)^2\mathrm{A}^2}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2} - \frac{2(1-\gamma)|c_1|(1-\beta)3(2\alpha+1)\mu\mathrm{B}}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2} + \frac{2|c_1|^2(1-\beta)^2(\alpha+1)^2\mathrm{A}^2}{4(\alpha+1)^2\mathrm{A}^2} -$$
$$\frac{|c_1|^2(1-\beta)^2 3(2\alpha+1)\mu\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) - \frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} -$$
$$\frac{3(2\alpha+1)\mu|c_1|^2(1-\beta)^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2} + \frac{2(1-\gamma)|c_1|(1-\beta)(\alpha+1)^2\mathrm{A}^2}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2} - \frac{3(2\alpha+1)(1-\gamma)(1-\beta)|c_1|\mu\mathrm{B}}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) - \frac{3(2\alpha+1)(1-\gamma)^2\mathrm{B}\mu}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} -$$
$$\frac{3(2\alpha+1)\mu|c_1|^2(1-\beta)^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2} + \frac{(1-\gamma)(1-\beta)\big(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}\big)|c_1|}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}, \tag{3.20}$$

$$= \mathcal{Q}(x)\,; x = |c_1|.$$

We defined x

$$\mathcal{Q}(x) = x^2 - \frac{4(1-\gamma)\big(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}\big)}{3(2\alpha+1)\mu(1-\beta)(w_2-d_2)\mathrm{B}}x - \frac{16(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2 d_2}{3(w_3-d_3)(w_2-d_2)(2\alpha+1)(1-\beta)^2\mu\mathrm{B}} -$$
$$\frac{8(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(w_3-d_3)(2\alpha+1)(1-\beta)^2\mu\mathrm{B}} - \frac{8(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(1-\beta)\mu\mathrm{B}} + \frac{4(1-\gamma)^2}{(w_2-d_2)^2(1-\beta)^2},$$

After doing some operations, we get

$$x=\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}.$$

Now, substituting in (3.20)

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)\mu(1-\gamma)^2\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}-$$

$$\frac{,3(2\alpha+1)\mu(1-\beta)^2\ \mathrm{B}\left(\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}\right)^2}{4(\alpha+1)^2\mathrm{A}^2}+$$

$$\frac{(1-\gamma)(1-\beta)(\ 2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})\left(\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}\right)}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}-$$

$$\frac{4(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{(4)3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}}+\frac{2(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}+$$

$$\frac{(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}+$$

$$\frac{(1-\gamma)^2\ (4(\alpha+1)^4\mathrm{A}^4-12(\alpha+1)^2(2\alpha+1)\mathrm{A}^2\mu\mathrm{B}+9(2\alpha+1)^2\mu^2\mathrm{B}^2)}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}+$$

$$\frac{4(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\leq\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{4(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}}-$$

$$\frac{4(1-\gamma)^2}{(w_2-d_2)^2}\ , \qquad (3.21)$$

Now $\lceil x\rceil\leq 2$ we get the interval

$$\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}\leq 2$$

$$\frac{4(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{6(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}\leq\mu\,,$$

then

$$\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} \le \mu, \qquad (3.22)$$

hence result (3.21) concludes for the case

$$\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} \le \mu \quad \le \frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}.$$

Second, consider the case $\mu \le \mu_0$

$$\mu \le \mu_0 = \frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)},$$

Write :

$$a_3-\mu a_2^2 = a_3-\mu_0 a_2^2+\mu_0 a_2^2-\mu a_2^2$$

$$|a_3-\mu a_2^2| \le |a_3-\mu_0 a_2^2| + |\mu_0-\mu||a_2^2|, \qquad (3.23)$$

From Theorem 3.2.1, we obtain:

$$|a_2| \le \frac{(1-\ \beta)}{(\alpha+1)\mathrm{A}} + \frac{(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}} \Rightarrow |a_2| \le \frac{(w_2-d_2)(1-\ \beta)+(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}}, \quad (3.24)$$

From (3.23), substituting $|a_3-\mu a_2^2|$,

Then, we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le 3(2\alpha+1)\mathrm{B}|a_3-\mu_0 a_2^2| + 3(2\alpha+1)\mathrm{B}|\mu_0 - \mu||a_2^2|, \qquad (3.25)$$

From (3.23) substituting $|a_3-\mu a_2^2|$, and from (3.21) substituting

$3(2\alpha+1)\mathrm{B}|a_3-\mu_0 a_2^2|$, and from (3.24) substituting $|a_2^2|$,and, we have

$$|\mu_0-\mu| = \left|\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} - \mu\right|$$

$$|\mu_0-\mu| = \left|\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\right.$$

$$\left. - \frac{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\right|,$$

because Positive $0 \le \alpha < 1 <, 0 \le \beta < 1, 0 \le \gamma < 1 <,\ w_2 \ge d_2,$

$$|\mu_0-\mu| = \frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\,. \tag{3.26}$$

Now substituting in (3.25), we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) - \frac{4(1-\gamma)^2}{(w_2-d_2)^2} +$$

$$\frac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}} + 3(2\alpha +$$

$$1)\mathrm{B}\left(\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\right)\left(\frac{(w_2-d_2)(1-\beta)+(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}}\right)^2,$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) +$$

$$\frac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mathrm{B}}\left(\frac{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}\right) - \frac{4(1-\gamma)^2}{(w_2-d_2)^2} +$$

$$\left(\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}\right)\big((w_2-d_2)(1-\beta)+(1-\gamma)\big),$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\begin{aligned}
&\le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta)\\
&+ \frac{2(1-\gamma)\big((w_2-d_2)(1-\beta)+(1-\gamma)\big)}{(w_2-d_2)^2} - \frac{4(1-\gamma)^2}{(w_2-d_2)^2}\\
&+ \frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}\\
&- \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},
\end{aligned}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2 \le |\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)$$

$$+\frac{2(1-\gamma)\big((w_2-d_2)(1-\ \beta)+(1-\gamma)\big)}{(w_2-d_2)^2}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}$$

$$+\frac{2(1-\gamma)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{(w_2-d_2)^2}$$

$$-\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\le\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}$$

$$+\frac{2(1-\gamma)(w_2-d_2)(1-\ \beta)}{(w_2-d_2)^2}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}$$

$$+\frac{2(1-\gamma)(w_2-d_2)(1-\beta)}{(w_2-d_2)^2}$$

$$-\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\le\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{4(1-\gamma)(1-\ \beta)}{(w_2-d_2)}$$

$$-\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}. \qquad (\ 3.27)$$

Consider

$$\mu=\mu_1=\frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}, \qquad (3.28)$$

substituting in (3.21), we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}$$

$$+\frac{(3)4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2\ (2\alpha+1)\mathrm{B}}{(2)3(2\alpha+1)(w_2-d_2)^2\mathrm{B}\ (\alpha+1)^2\mathrm{A}^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|$$

$$\leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}$$

$$-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}. \qquad (3.29)$$

Now case $\mu_2 < \mu$,

$$a_3-\mu a_2^2 = a_3-\mu_2 a_2^2+\mu_2 a_2^2-\mu a_2^2 = a_3-\mu_2 a_2^2+(\mu_2-\mu)a_2^2 \qquad (3.30)$$

$$|a_2|^2 \leq \frac{\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}.$$

Defined condition $\mu_2 < \mu$,

$$\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2} \leq -\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}-\frac{2(1-\gamma)}{w_3-d_3}-$$

$$2(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)}+\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

we put μ in the right tip:

$$\frac{8(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{4(1-\gamma)}{w_3-d_3}+4(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}$$

$$+\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)}$$

$$\leq \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

we divide all terms by a coefficient, we get

$$\begin{aligned}
&\frac{8(1-\gamma)^2(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2d_2}{3(2\alpha+1)(w_2-d_2)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad-\frac{2(1-\gamma)^2(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\gamma)(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\le\mu
\end{aligned}$$

$$\begin{aligned}
&\frac{8(1-\gamma)^2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2d_2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad-\frac{2(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\\
&\quad+\frac{4(1-\gamma)(1-\beta)(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}\le\mu.
\end{aligned}$$

Now substituting in this inequality

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2|\le 3(2\alpha+1)\mathrm{B}|a_3-\mu_2a_2^2|+3(2\alpha+1)\mathrm{B}|\mu_2-\mu||a_2|^2,$$

From (3.28) and we multiply this $|\mu_2-\mu|$ by the negative

$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}+$

$3(2\alpha+1)\mathrm{B}\Big(-\frac{8(1-\gamma)^2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2 d_2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}-$

$\frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}-\frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}+$

$\frac{2(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}-\frac{4(1-\gamma)(1-\beta)(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}+$

$\mu\Big)\left(\frac{\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}\right),$

we get

$$\begin{aligned}
&3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \\
&\quad\leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2} \\
&\quad-\frac{8(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}-\frac{4(1-\gamma)}{(w_3-d_3)}-4(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)} \\
&\quad+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}+\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},
\end{aligned}$$

then

$$\begin{aligned}
&3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \\
&\quad\leq -\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}-\frac{2(1-\gamma)}{w_3-d_3}-2(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)} \\
&\quad+\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}. \quad \blacksquare \qquad (3.31)
\end{aligned}$$

Corollary 3.2.1

Let f given by (1.1) be in the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$ where $\lambda_2=0$, $m=1, n=0$ .Then. We get,

$$3(2\alpha+1)|a_3-\mu a_2^2|$$

$$\leq \begin{cases} \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)}+\frac{4(1-\gamma)(1-\ \beta)}{k_2-d_2}+\frac{2(1-\gamma)}{k_3-d_3}+2(1-\ \beta)-\frac{2\mu\left(1-\gamma+(1-\beta)(k_2-d_2)\right)^2}{\mu_1(k_2-d_2)^2} \ if\ \mu \leq \mu_0\,, \\ \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)}-\frac{4(1-\gamma)^2}{(k_2-d_2)^2}+\frac{2(1-\gamma)}{k_3-d_3}+2(1-\ \beta)+\frac{2\mu_1(1-\gamma)^2}{(k_2-d_2)^2\mu} \ if\ \mu_0 \leq \mu \leq \mu_1, \\ \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)}-\frac{2(1-\gamma)^2}{(k_2-d_2)^2}+\frac{2(1-\gamma)}{k_3-d_3}+2(1-\ \beta) \quad if\ \mu_1 \leq \mu \leq \mu_2\,, \\ -\frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)}-\frac{4(1-\gamma)(1-\ \beta)}{k_2-d_2}-\frac{2(1-\gamma)}{k_3-d_3}-2(1-\ \beta)+\frac{2\mu\left(1-\gamma+(1-\beta)(k_2-d_2)\right)^2}{\mu_1(k_2-d_2)^2} \ if\ \mu_2 \leq \mu\,, \end{cases}$$

where

$$\mu_0=\frac{\mu_1(1-\gamma)}{1-\gamma+(1-\beta)(k_2-d_2)}\,,$$

$$\mu_1=\frac{2(\alpha+1)^2}{3(2\alpha+1)}\ ,$$

$$\mu_2=\frac{\mu_0(k_2-d_2)^2}{1-\gamma+(1-\beta)(k_2-d_2)}\left(\frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)}+\frac{2}{k_3-d_3}+\frac{2(1-\ \beta)}{(1-\gamma)}+\frac{2(1-\ \beta)}{(k_2-d_2)}-\frac{(1-\gamma)}{(k_2-d_2)^2}\right).$$

These results are based on the reference.(Alharayzeh & Darus, 2022)

Chapter Four

Certain Subclasses of Bi-Univalent Functions Related to Generalized Derivative Operator

4.1 Introduction.

(Lewin, 1967) investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Later(Brannan & Clunie, 1980) Conjectured that $|a_2| < \sqrt{2}$. Subsequently,(Netanyahu, 1969) on the other hand, showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$. To explore various interesting examples of functions in the class Σ, refer to the pioneering work on this subject by Srivastava et al (Srivastava et al., 2010) .(Taha & BRANNAN, 1986) introduced certain subclasses of the bi-univalent function class Σ, similar to the familiar subclasses $S^*(\beta)$ and $\mathcal{C}(\beta)$ of univalent function S. The classes $S_\Sigma^*(\beta)$ and $\mathcal{C}_\Sigma(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β.

However, The general coefficient estimate bounds on $|a_n|$ $(n \in \{4,5,6,...\})$for a function $f \in \Sigma$ defined by (1.1) remain an unsolved problem. (Motamednezhad et al., 2022) The determination of estimates for the Tayler-Maclaurin coefficients $a_n$is an important concern problem in geometric function theory as it provides information a bout the geometric properties of these functions. Motivated by the aforementioned works and making use of the generalized derivative operator, we investigate two subclasses of analytic and bi-univalent functions using the techniques employed by (Frasin & Aouf, 2011; Sabir, 2024; Srivastava et al., 2010). The obtained results improve and generalize some recent works and rectify remarkable mistakes in existing coefficient estimates.

We denote by S the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$. For example, the Koebe function is in S,

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k, \quad (z \in \mathbb{U}).$$

It is well known that the image of $\mathbb{U}$ under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1}such that

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots. \quad (4.1)$$

The family of all bi-univalent functions in unit disk given by (1.1) is denoted by Σ. (Brannan & Taha, 1988) introduced some bi-univalent function class Σ subclasses that are similar to the famous subclasses $S^*(\beta)$ and $\mathcal{C}(\beta)$ of starlike and convex functions of order β ($0 \leq \beta < 1$), respectively. Thus, a function $f(z) \in \Sigma$ is in the class $S_\Sigma^*(\beta)$ of bi-starlike functions of order β if both f and f^{-1} are starlike functions of order β $(0 < \beta \leq 1)$, or into the class $\mathcal{C}_\Sigma(\beta)$ of bi-convex functions of order β if both f and f^{-1} are convex functions of order β $(o < \beta \leq 1)$. Moreover, a function $f(z) \in \mathcal{A}$ is bi-starlike functions of order β, denoted by the class $S_\Sigma^*(\beta)$, if it satisfies each of the following conditions:

$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}$ $(0 < \beta \leq 1, z \in \mathbb{U})$ and $\left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\beta\pi}{2}$ $(0 < \beta \leq 1,$ $w \in \mathbb{U})$, where g is the extension of $f^{-1}(z)$ to $\mathbb{U}$.(Sabir, 2024)

This chapter aims to find the coefficients estimate $|a_2|$ and $|a_3|$ for functions in two new subclasses of the function $f(z) \in \Sigma$.

To conclude the results, we need Lemma 1.4.3.1 to prove the theorems.

4.2 Coefficient Bounds for Functions in the Class $\mathcal{L}_\Sigma(\eta, \omega, m, \alpha)$.

Let $\mathcal{C}(z)$ be any complex-valued function and function $p \in \mathcal{P}$ given by Definition 1.4.3.1 such that

$$|\arg(\mathcal{C}(z))| = \alpha|\arg(p(z)| < \frac{\alpha\pi}{2}.$$

If $|\arg(\mathcal{C}(z))| < \frac{\alpha\pi}{2}$, we can express the function $\mathcal{C}(z)$ in terms of p and α as follows:

$$\mathcal{C}(z) = [\mathcal{P}(z)]^\alpha, 0 < \alpha \leq 1.$$ (Sabir, 2024)

Definition 4.2.1

Let $(\eta \geq 0\,, \omega \in \mathbb{C}\backslash\{0\}\,, m \in \mathbb{N}_0 \ and \ 0 < \ \alpha \leq 1)$. We say that a function $f(z)$ given by (1.1) is in the class $\mathcal{L}_\Sigma(\eta, \omega, m, \alpha)$ if the following conditions are satisfied:

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}\ , z \in \mathbb{U}, \qquad (4.2)$$

and

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}\ , w \in \mathbb{U} \qquad (4.3)$$

where the function $g = f^{-1}$ is defined by (4.1).

Marks:

$$\mathrm{A} = \frac{(1+\lambda_1+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2)^m}(n+1)\ \ , \mathrm{B} = \frac{(1+\lambda_1(2)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(2)\right)^m}\frac{(n+2)(n+1)}{2}.$$

Theorem 4.2.1

Let $f \in \mathcal{L}_\Sigma(\eta, \omega, m, \alpha)$ be given by (1.1). Then

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{|2\alpha\omega(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2)-(\alpha-1)(1+\eta)^2\mathrm{A}^2|}}\,, \qquad (4.4)$$

and

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2\mathrm{A}^2} + \frac{\alpha|\omega|}{(1+2\eta)\mathrm{B}}. \qquad (4.5)$$

Proof

From the Definition 4.2.1 we get

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right] = [h(z)]^\alpha, \qquad (4.6)$$

and

$$1+\frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)}+\eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'+w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}-1\right]=$$

$$[q(w)]^{\alpha}, \tag{4.7}$$

where $h, q \in \mathcal{P}$ have the formula as follows

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \tag{4.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots, \tag{4.9}$$

then

$$[h(z)]^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k} 1^{\alpha-k}\left(\sum_{n=1}^{\infty} h_n z^n\right)^k$$

$$[h(z)]^{\alpha} = \binom{\alpha}{0} 1^{\alpha} + \binom{\alpha}{1} 1^{\alpha-1}\sum_{n=1}^{\infty} h_n z^n + \binom{\alpha}{2} 1^{\alpha-2}\left(\sum_{n=1}^{\infty} h_n z^n\right)^2 +$$

$$\binom{\alpha}{3} 1^{\alpha-3}\left(\sum_{n=1}^{\infty} h_n z^n\right)^3 + \cdots$$

$$[h(z)]^{\alpha} = \binom{\alpha}{0} 1^{\alpha} + \binom{\alpha}{1} 1^{\alpha-1}(h_1 z + h_2 z^2 + h_3 z^3 + \cdots) + \binom{\alpha}{2} 1^{\alpha-2}(h_1^2 z^2 +$$

$$h_1 h_2 z^3 + \cdots + h_1 h_2 z^3 + \cdots) + \cdots$$

$$[h(z)]^{\alpha} = 1 + \alpha h_1 z + \alpha h_2 z^2 + \alpha h_3 z^3 + \cdots + \frac{\alpha(\alpha-1)}{2!} h_1^2 z^2 + \frac{\alpha(\alpha-1)}{2!} h_1 h_2 z^3 + \cdots +$$

$$\frac{\alpha(\alpha-1)}{2!} h_1 h_2 z^3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^3 z^3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^2 h_2 z^4 + \cdots$$

$$[h(z)]^{\alpha} = 1 + \alpha h_1 z + \left[\alpha h_2 + \frac{\alpha(\alpha-1)}{2!} h_1^2\right] z^2 + \left[\alpha h_3 + \frac{2\alpha(\alpha-1)}{2!} h_1 h_2 +\right.$$

$$\left.\frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^3\right] z^3 + \cdots$$

$$[h(z)]^{\alpha} = 1 + \alpha h_1 z + \left[\alpha h_2 + \frac{\alpha(\alpha-1)}{2!} h_1^2\right] z^2 + \left[\alpha h_3 + \alpha(\alpha-1) h_1 h_2 +\right.$$

$$\left.\frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^3\right] z^3 + \cdots, \tag{4.10}$$

and

$$[q(w)]^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k} 1^{\alpha-k}\left(\sum_{n=1}^{\infty} q_n w^n\right)^k$$

$$[q(w)]^{\alpha} = \binom{\alpha}{0} 1^{\alpha} + \binom{\alpha}{1} 1^{\alpha-1}\sum_{n=1}^{\infty} q_n w^n + \binom{\alpha}{2} 1^{\alpha-2}\left(\sum_{n=1}^{\infty} q_n w^n\right)^2 +$$

$$\binom{\alpha}{3} 1^{\alpha-3}\left(\sum_{n=1}^{\infty} q_n w^n\right)^3 + \cdots$$

$$[q(w)]^{\alpha} = 1 + \alpha q_1 w + \left[\alpha q_2 + \frac{\alpha(\alpha-1)}{2!} q_1^2\right] w^2 + \left[\alpha q_3 + \frac{2\alpha(\alpha-1)}{2!} q_1 q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} q_1^3\right] w^3 + \cdots$$

$$[q(w)]^{\alpha} = 1 + \alpha q_1 w + \left[\alpha q_2 + \frac{\alpha(\alpha-1)}{2!} q_1^2\right] w^2 + \left[\alpha q_3 + \alpha(\alpha-1) q_1 q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} q_1^3\right] w^3 + \cdots, \tag{4.11}$$

when substituting in the class we get

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\left(1 + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right) - 1\right] = 1 + \frac{1}{\omega}\ [1 - \eta + (1-\eta)\mathrm{A}\, a_2 z + (1-\eta)(2\mathrm{B}a_3 - \mathrm{A}^2 a_2^2)\, z^2 + \cdots + \eta + 2\eta\mathrm{A}\, a_2 z + \eta(6\mathrm{B}a_3 - 4\mathrm{A}^2 a_2^2)\, z^2\ + \cdots - 1]$$

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\left(1 + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right) - 1\right] = 1 + \frac{(1+\eta)}{\omega}\mathrm{A}\, a_2 z + \left[\frac{2(1+2\eta)\,\mathrm{B}}{\omega} a_3 - \frac{(1+3\eta)}{\omega}\mathrm{A}^2\, a_2^2\right] z^2\ + \cdots. \tag{4.12}$$

Now, we defined the inverse

$$I^m(\lambda_1,\lambda_2,l,n)g(w) = w - \mathrm{A}\, a_2 w^2 + \mathrm{B}(2a_2^2 - a_3) w^3 - \cdots.$$

After performing some operations, we obtain:

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\left(\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right) - 1\right] = 1 + \frac{1}{\omega}[1 - \eta - (1-\eta)\mathrm{A}a_2 w + (1-\eta)(2\mathrm{B}(2a_2^2 - a_3) - \mathrm{A}^2 a_2^2) w^2 + \cdots + \eta - 2\eta\mathrm{A}\, a_2 w + \eta(6\mathrm{B}(2a_2^2 - a_3) - 4\mathrm{A}^2 a_2^2) w^2 + \cdots - 1]$$

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\left(\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right) - 1\right] = 1 + \frac{1}{\omega}[-(1+\eta)\mathrm{A}a_2 w + (4(1+2\eta)\mathrm{B}a_2^2 - 2(1+2\eta)\mathrm{B}a_3 - (1+3\eta)\mathrm{A}^2\, a_2^2) w^2 + \cdots]$$

$$1+\frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^{m}(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^{m}(\lambda_1,\lambda_2,l,n)g(w)}+\eta\left(\frac{\left(I^{m}(\lambda_1,\lambda_2,l,n)g(w)\right)'+z\left(I^{m}(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^{m}(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right)-1\right]=$$

$$1-\frac{(1+\eta)}{\omega}\mathrm{A}a_2 w+\left[\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)\mathrm{A}^2}{\omega}\ a_2^2\right]w^2+\cdots. \quad (4.13)$$

Coefficients equal to (4.10), (4.12)

$$\frac{(1+\eta)}{\omega}\mathrm{A}a_2=\alpha h_1 \qquad (4.14)$$

$$\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)}{\omega}\ \mathrm{A}^2a_2^2=\alpha h_2+\frac{\alpha(\alpha-1)}{2!}h_1^2, \qquad (4.15)$$

from (4.11), (4.13) we get

$$-\frac{(1+\eta)}{\omega}\mathrm{A}a_2=\alpha q_1 \qquad (4.16)$$

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)\mathrm{A}^2}{\omega}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2!}q_1^2, \qquad (4.17)$$

Clearly, from (4.14), (4.16) we have

$$h_1=-q_1, \qquad (4.18)$$

we square and addition (4.14), (4.16) together ,we get

$$\frac{2(1+\eta)^2\mathrm{A}^2}{\omega^2}a_2^2=\alpha^2(h_1^2+q_1^2), \qquad (4.19)$$

we addition (4.15), (4.17) we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+3\eta)\mathrm{A}^2}{\omega}a_2^2=\frac{1}{2}\alpha(\alpha-1)(h_1^2+q_1^2)+\alpha(h_2+q_2), \qquad (4.20)$$

from (4.19) substituting$(h_1^2+q_1^2)$ in (4.20)

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+3\eta)\mathrm{A}^2}{\omega}a_2^2-\frac{1}{2}\alpha(\alpha-1)\left[\frac{2(1+\eta)^2\mathrm{A}^2}{\alpha^2\omega^2}a_2^2\right]=\alpha(h_2+q_2),$$

now

$$\frac{4\alpha\omega(1+2\eta)\mathrm{B}}{\alpha\omega^2}a_2^2-\frac{2\alpha\omega(1+3\eta)\mathrm{A}^2}{\alpha\omega^2}a_2^2-\frac{(\alpha-1)(1+\eta)^2\mathrm{A}^2}{\alpha\omega^2}a_2^2=\alpha(h_2+q_2),$$

$$a_2^2\left[\frac{\alpha\omega(4(1+2\eta)\mathrm{B}-2(1+3\eta)\mathrm{A}^2)-(\alpha-1)(1+\eta)^2\mathrm{A}^2}{\alpha\omega^2}\right]=\alpha(h_2+q_2)$$

$$a_2^2=\frac{\alpha^2\omega^2(h_2+q_2)}{2\alpha\omega(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2)-(\alpha-1)(1+\eta)^2\mathrm{A}^2}, \qquad (4.21)$$

from Lemma 1.4.3.1 for coefficients h_2 and $q_2$on (4.21) imply that

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{|2\alpha\omega(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2)-(\alpha-1)(1+\eta)^2\mathrm{A}^2|}}.$$

Next, we exist $|a_3|$, by subtracting (4.17) from (4.15), we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_3 - \frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2 = \frac{1}{2}\alpha(\alpha-1)(h_1^2-q_1^2)+\alpha(h_2-q_2), \tag{4.22}$$

from (4.19) we substituting a_2^2 in (4.22) and we put $h_1^2 = q_1^2$ from (4.18)

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_3 - \frac{4(1+2\eta)\mathrm{B}}{\omega}\left[\frac{\alpha^2\omega^2\left(h_1^2+q_1^2\right)}{2(1+\eta)^2\mathrm{A}^2}\right] = \frac{1}{2}\alpha(\alpha-1)(h_1^2-h_1^2)+\alpha(h_2-q_2),$$

$$a_3 = \frac{\alpha^2\omega^2\left(h_1^2+q_1^2\right)}{2(1+\eta)^2\mathrm{A}^2}+\frac{\alpha\omega(h_2-q_2)}{4(1+2\eta)\mathrm{B}}. \tag{4.23}$$

Now, applying Lemma 1.4.3.1 for coefficients h_1, h_2, q_1 and q_2 on (4.23) we get

$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2\mathrm{A}^2}+\frac{\alpha|\omega|}{(1+2\eta)\mathrm{B}}$. ■

Corollary 4.2.1

If $m = 1$, $\lambda_2 = 0$ .Let f given by (1.1) be in the class $\mathcal{L}_\Sigma(\eta,\omega,\ell,\alpha)$. Then

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{\left|2\alpha\omega\left((1+2\eta)(\ell+2)(\ell+1)-(1+3\eta)(\ell+1)^2\right)-(\alpha-1)(1+\eta)^2(\ell+1)^2\right|}},$$

and

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2(\ell+1)^2}+\frac{2\alpha|\omega|}{(1+2\eta)(\ell+2)(\ell+1)}.$$

These results are based on the reference.(Sabir, 2024)

4.3 Coefficient Bounds for the Functions in the Class $\mathcal{L}_\Sigma^*(\eta,\omega,m,\beta)$.

Let $\mathcal{H}(z)$ be any complex-valued function and function $p \in \mathcal{P}$ given by Definition 1.4.3.1 such that

$$Re\,\{\mathcal{H}(z)\} = \beta + (1-\beta)Re\,\{p(z)\} > \beta.$$

If $Re\,\{\mathcal{H}(z)\} > \beta$, we can express the function $\mathcal{H}(z)$ in terms of p and β as follows:

$\mathcal{H}(z) = \beta + (1-\beta)p(z)$, $0 \leq \beta < 1$.(Sabir, 2024)

Definition 4.3.1

Let $(\eta \geq 0, \omega \in \mathbb{C} \setminus \{0\}, m \in \mathbb{N}_0 \ and, \ 0 \ \leq \beta < 1)$. We say that a function $f(z)$ given by (1.1) is in the class $\mathcal{L}_{\Sigma}^{*}(\eta, \omega, m, \beta)$ if the following conditions are satisfied:

$$Re \left\{1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - 1\right]\right\} > \beta, z \in \mathbb{U}, \tag{4.24}$$

and

$$Re \left\{1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right]\right\} > \beta, w \in \mathbb{U}, \tag{4.25}$$

where the function $g = f^{-1}$ is defined by (4.1).

Theorem 4.3.1

Let $f \in \mathcal{L}_{\Sigma}^{*}(\eta, \omega, m, \beta)$ be given by (1.1). Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|}}, & 0 \leq \beta \leq 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega||2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} \\ \frac{2|\omega|(1-\beta)}{(1+\eta)\mathrm{A}}, & 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega||2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} \leq \beta < 1, \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}}, & 0 \leq \beta \leq 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega||2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} \\ \frac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}}, & 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega||2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} \leq \beta < 1. \end{cases}$$

Proof

From the definition we get

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - 1\right] = \beta + (1-\beta)h(z), \tag{4.26}$$

and

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right] =$$

$$\beta + (1-\beta)q(z), \tag{4.27}$$

where $h, q \in \mathcal{P}$.Then

$$\begin{aligned}\beta + (1-\beta)h(z) &= \beta + (1-\beta)[1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots] \\ &= \beta + 1 - \beta + (1-\beta)h_1 z + (1-\beta)h_2 z^2 + \cdots \\ &= 1 + (1-\beta)h_1 z + (1-\beta)h_2 z^2 + \cdots, \end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\beta + (1-\beta)q(z) &= \beta + (1-\beta)[1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots] \\ &= \beta + 1 - \beta + (1-\beta)q_1 z + (1-\beta)q_2 z^2 + \cdots \\ &= 1 + (1-\beta)q_1 z + (1-\beta)q_2 z^2 + \cdots. \end{aligned} \tag{4.29}$$

Now, equal coefficients (4.12) with (4.28) for h

$$\frac{(1+\eta)\mathrm{A}}{\omega} a_2 = (1-\beta)h_1 \tag{4.30}$$

$$\frac{2(1+2\eta)\mathrm{B}}{\omega} a_3 - \frac{(1+3\eta)\mathrm{A}^2}{\omega}\, a_2^2 = (1-\beta)h_2, \tag{4.31}$$

equal coefficients (4.13) with (4.29) for q

$$-\frac{(1+\eta)\mathrm{A}}{\omega} a_2 = (1-\beta)q_1 \tag{4.32}$$

$$\frac{4(1+2\eta)\mathrm{B}}{\omega} a_2^2 - \frac{(1+3\eta)\mathrm{A}^2}{\omega} a_2^2 - \frac{2(1+2\eta)\mathrm{B}}{\omega} a_3 = (1-\beta)q_2, \tag{4.33}$$

From (4.30) and (4.32) we get

$$h_1 = -q_{1,} \tag{4.34}$$

We square (4.30) and (4.32) then addition we get

$$\frac{2(1+\eta)^2\mathrm{A}^2}{\omega^2} a_2^2 = (1-\beta)^2\left({h_1}^2 + {q_1}^2\right), \tag{4.35}$$

we add (4.31) with (4.33) we get

$$\left[\frac{4(1+2\eta)\mathrm{B}}{\omega} - \frac{2(1+3\eta)\mathrm{A}^2}{\omega}\right] a_2^2 = (1-\beta)(h_1 + q_1), \tag{4.36}$$

from (4.35) we defined

$$a_2^2 = \frac{\omega^2(1-\beta)^2\left({h_1}^2 + {q_1}^2\right)}{2(1+\eta)^2\mathrm{A}^2}, \tag{4.37}$$

and from (4.36) we get

$$a_2^2=\frac{\omega(1-\beta)(h_2+q_2)}{4(1+2\eta)\mathrm{B}-2(1+3\eta)\mathrm{A}^2}\,, \tag{4.38}$$

the equations (4.37) and (4.38) together with applying Lemma 1.4.3.1 for the coefficients h_1, q_1, h_2, and q_2, we find that

$$a_2^2=\frac{4\omega^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2}\Rightarrow |a_2|\le\frac{2|\omega|(1-\beta)}{(1+\eta)\mathrm{A}},$$

and

$$a_2^2=\frac{4\omega(1-\beta)}{4(1+2\eta)\mathrm{B}-2(1+3\eta)\mathrm{A}^2}\Rightarrow a_2^2=\frac{4\omega(1-\beta)}{2(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2)}$$

$$a_2^2=\frac{2\omega(1-\beta)}{2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2}$$

$$|a_2|\le\sqrt{\frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|}}\,,$$

respectively.

To determine the estimates on $|a_3|$, by subtracting (4.33) from (4.31), we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2=(1-\beta)(h_2-q_2)$$

$$a_3=a_2^2+\frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}}\,. \tag{4.39}$$

Substituting the value of a_2^2 from (4.37) into (4.39) we get

$$a_3=\frac{\omega^2(1-\beta)^2\left({h_1}^2+{q_1}^2\right)}{2(1+\eta)^2\mathrm{A}^2}+\frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}}\,. \tag{4.40}$$

Substituting the value of a_2^2 from (4.38) into (4.39), we get

$$a_3=\frac{\omega(1-\beta)(h_2+q_2)}{4(1+2\eta)\mathrm{B}\,-2(1+3\eta)\mathrm{A}^2}+\frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}}, \tag{4.41}$$

respectively.

Finally, applying Lemma 1.4.3.1 for the coefficients h_2, h_2, q_1, and q_2 on equations (4.40) and (4.41) together, we conclude that

$$|a_3| \leq \frac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}},$$

and

$$|a_3| \leq \frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2|} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}}. \qquad \blacksquare$$

Corollary 4.3.1

If $m = 1$, $\lambda_2 = 0$ .Let f given by (1.1) be in the class $\mathcal{L}_{\Sigma}^{*}(\eta, \omega, n, \beta)$. Then

$|a_2| \leq$

$$\begin{cases} \sqrt{\dfrac{2|\omega|(1-\beta)}{|(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|}}, & 0 \leq \beta \leq 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \\ \dfrac{2|\omega|(1-\beta)}{(1+\eta)(n+1)} \ , & 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \leq \beta < 1 \end{cases}$$

and

$|a_3| \leq$

$$\begin{cases} \dfrac{2|\omega|(1-\beta)}{|(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} + \dfrac{2|\omega|(1-\beta)}{(1+2\eta)(n+2)(n+1)}, & 0 \leq \beta \leq 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \\ \dfrac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2(n+1)^2} + \dfrac{2|\omega|(1-\beta)}{(1+2\eta)(n+2)(n+1)} \ , & 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \leq \beta < 1 \end{cases}$$

These results are based on the reference. (Sabir, 2024)

Conclusion

In this study, we presented a generalized derivative operator for univalent and normalized functions in the open unit disk, we discussed geometric properties, and we studied the Fekete-Szego inequality on analytic univalent functions. It also introduced two new subclasses of analytic and bi-univalent functions, resulting in estimates of coefficient bounds.

The Recommendations

The results will be extended to the function

$$f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \ ; p \in \mathbb{N} = \{1,2,3, \dots \}$$

within the unit disk.

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